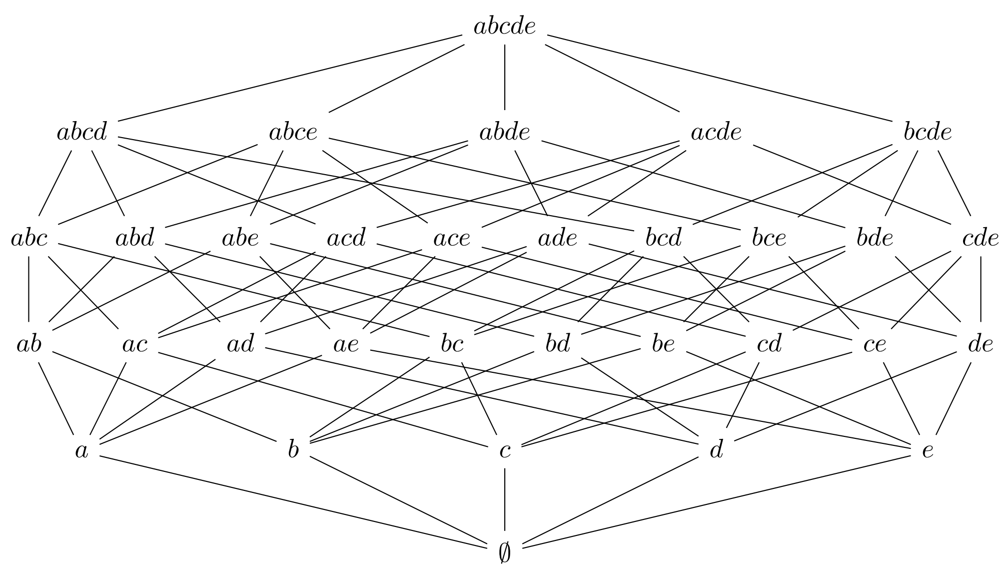
Read: Chapter 6 of *Algebraic and Combinatorial Computational Biology* (Robeva & Macauley, 2018). Inferring interactions in molecular networks via primary decompositions of monomial ideals. By M. Macauley and B. Stigler, pages 175–210.

Chapter 3 of *Mathematical Concepts and Methods in Modern Biology* (Robeva & Hodge, 2013). Inferring the topology of gene regulatory networks: an algebraic approach to reverse engineering. By B. Stigler and E. Dimitrova, pages 75–90.

Exercises.

1. Consider the simplicial complex $\Delta = \{\emptyset, a, b, c, d, e, ab, ac, ad, bc, cd, ce, abc\}$ over the set $X = \{a, b, c, d, e\}$. The 5-dimensional Boolean lattice 2^X is shown below.
   (i) Sketch the simplicial complex and find the maximal faces.
   (ii) Draw a dashed or colored line that "cuts" the Boolean lattice into two halfs – below the line are the faces and above are the non-faces.
   (iii) Circle each node in the Boolean lattice 2^X corresponding to a maximal face of Δ.
   (iv) Box each node that corresponds to a minimal non-face.
   (v) Find the Stanley-Reisner ideal I_{Δ^c} and compute its primary decomposition.



2. Consider a Boolean network $f = (f_1, f_2, f_3)$ whose (synchronous) phase spaces consists of the following:

$$(0,0,1) \xrightarrow{f} (1,0,1) \xrightarrow{f} (1,1,1) \xrightarrow{f} (1,1,0) \xrightarrow{f} (0,1,0) \xrightarrow{f} (0,0,0).$$

   In this problem, you will reverse-engineer the wiring diagram.

   (a) Do the following steps for each $k = 1, 2, 3$.

i. Write down the corresponding set of *data*

$$\mathcal{D}_k := \{(\mathbf{s}_1, t_{1k}), \dots, (\mathbf{s}_5, t_{5k})\}$$

that arises from the k^{th} coordinate of this time-series.

ii. Compute the monomials $m(\mathbf{s}_i, \mathbf{s}_j)$ for which $t_{ik} \neq t_{jk}$, and find the ideal $I_{\Delta_k^c}$ of non-disposable sets.

iii. Find the primary decomposition of $I_{\Delta_k^c}$.

iv. Find all min-sets of $\mathcal{D}_k$, and sketch a wiring diagram for each. Only include edges incident to x_k.

(b) Repeat Steps (ii)–(iv) from Part (a) but to find the signed min-sets. That is, use the pseudomonomials $p(\mathbf{s}_i, \mathbf{s}_j)$ to compute the ideal $J_{\Delta_k^c}$ of signed non-disposable sets.

3. Consider the following *time series* of a 3-node algebra model over $\mathbb{F}_3$:

$$(1,1,1) \xrightarrow{f} (2,0,1) \xrightarrow{f} (2,0,0) \xrightarrow{f} (0,2,2) \xrightarrow{f} (0,2,2)\,.$$

For reference, here are the input vectors $\mathbf{s}_i$ and output vectors $\mathbf{t}_i$:

$$\begin{aligned}
\mathbf{s}_1 &= (s_{11}, s_{12}, s_{13}) = (1,1,1)\,, & \mathbf{t}_1 &= (t_{11}, t_{12}, t_{13}) = (2,0,1)\,,\\
\mathbf{s}_2 &= (s_{21}, s_{22}, s_{23}) = (2,0,1)\,, & \mathbf{t}_2 &= (t_{21}, t_{22}, t_{23}) = (2,0,0)\,,\\
\mathbf{s}_3 &= (s_{31}, s_{32}, s_{33}) = (2,0,0)\,, & \mathbf{t}_3 &= (t_{31}, t_{32}, t_{33}) = (0,2,2)\,,\\
\mathbf{s}_4 &= (s_{41}, s_{42}, s_{43}) = (0,2,2)\,, & \mathbf{t}_4 &= (t_{41}, t_{42}, t_{43}) = (0,2,2)\,.
\end{aligned}$$

(a) Find polynomials f_1, f_2, f_3 in $\mathbb{F}_3[x_1, x_2, x_3]/\langle x_1^3 - x_1, x_2^3 - x_2, x_3^3 - x_3\rangle$ that fit the data. That is, $f_j(\mathbf{s}_i) = t_{ij}$ for all $i = 1, 2, 3, 4$.

(b) For each $i = 1, 2, 3, 4$, write down the ideal $I_i = I(\mathbf{s}_i)$ of polynomials that vanish on the data point $\mathbf{s}_i$.

(c) Compute a Gröbner basis for the ideal I of polynomials that vanish on *all* of the input data points. You may use Sage, Singular, or Macaulay2, but use the graded reverse lexciographical monomial order `GRevLex` (this is the default).

(d) Write the *model space* of the time series using your answer to Part (a) as the particular solution.

(e) Compute the *normal form* of f_1, f_2, f_3 with respect to $\mathcal{G}$ by reducing them modulo the ideal I. Write the model space using this particular solution.

(f) Repeat Parts (c)–(e) using lex (`Lex`), and then graded lex (`GLex`).

(g) How big is the model space? How big is the vanishing ideal?