Exercises.

1. Let I be an ideal of a commutative ring R. Prove that R/I is a field if and only if I is maximal.

2. Prove that any finite integral domain is a field.

3. Let K be a finite field. The *characteristic* of K, denoted $\operatorname{char} K$, is the smallest positive integer n for which $n1 := \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$.

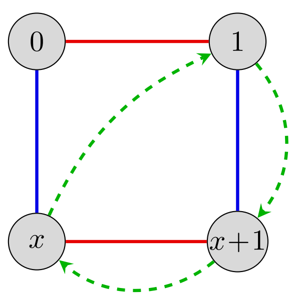
   (a) Prove that the characteristic of K is prime.

   (b) Show that K is a vector space over $\mathbb{F}_p$, where $p = \operatorname{char} K$.

   (c) Show that the order $|K|$ of K (the number of elements it contains) is a prime power.

   (d) Show that if K and L are finite fields with $K \subset L$ and $|K| = p^m$ and $|L| = p^n$, then m divides n.

4. The finite field $\mathbb{F}_4$ on 4 elements can be constructed as the quotient of the polynomial $\mathbb{F}_2[x]$ by the ideal $I = \langle x^2 + x + 1 \rangle$ generated by the irreducible polynomial $x^2 + x + 1$. The figure below shows a Cayley diagram, and multiplication and addition tables for the finite field $\mathbb{F}_2[x]/\langle x^2 + x + 1 \rangle \cong \mathbb{F}_4$.



$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

   (a) Find a degree-3 polynomial $f \in \mathbb{F}_2[x]$ that is irreducible over $\mathbb{F}_2$, and a degree-2 polynomial $g \in \mathbb{F}_3[x]$ that is irreducible over $\mathbb{F}_3$. [*Hint*: Any polynomial with no roots in the "prime field" $\mathbb{F}_p$ will work.]

   (b) Construct Cayley diagrams, addition, and multiplication tables for the finite fields

$$\mathbb{F}_8 \cong \mathbb{F}_2[x]/\langle f \rangle \qquad \text{and} \qquad \mathbb{F}_9 \cong \mathbb{F}_3[x]/\langle g \rangle\,.$$

5. Consider the following Boolean network (f_1, f_2, f_3):

$$f_{x_1} = x_1 \vee x_2, \qquad f_{x_2} = x_2 \wedge x_3, \qquad f_3 = \overline{x_1}.$$

   (a) Draw the wiring diagram, synchronous phase space, and asynchronous phase space.

   (b) Write down the ideal I of $\mathbb{F}_2[x_1, x_2, x_3]/\langle x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3 \rangle$ that describes a system of polynomials whose solutions are the fixed points.

(c) Use Sage or Macaulay2 to find a Gröbner basis of this ideal (with respect to "lex").

6. Recall our original 3-node Boolean model of the *lac* operon:

$$f_M = \overline{G_e} \wedge (L \vee L_e), \qquad f_E = M, \qquad f_L = \overline{G_e} \wedge ((E \wedge L_e) \vee (L \wedge \overline{E})).$$

For each of the 4 possible initial conditions, $(G_e, L_e) \in \mathbb{F}_2^2$, the model had one connected component with the biologically correct fixed point. Compute the probability that this would have happened purely by chance. (Assume a uniform distribution.)