MTHSC 208, WEEK 11

Week 10 summary

- Vector space: Set of Vectors X, and scalars ($\mathbb{R}$ or $\mathbb{C}$).
  - * Closed under addition
  - * Closed under scalar multiplication

A basis is a minimal set $\{X_1, \ldots, X_n\}$ such that every vector v can be written uniquely: $X = c_1 X_1 + c_2 X_2 + \ldots + c_n X_n$.

Ex: $\mathbb{R}^n$, polynomials of degree $\leq n$, power series periodic functions of period T, solutions of linear homog. ODEs.

- Solving ODEs with power series & generalized power series

Singular pts: Consider $y'' + P(x)y' + Q(x)y = 0$.

* x_0 is ordinary if "$P(x_0)$ and $Q(x_0)$ are defined" (almost)
* x_0 is singular otherwise. In this case:
  - x_0 is a regular singular point if $(x-x_0)P(x)$ & $(x-x_0)^2 Q(x_0)$ are defined* (almost)
  - x_0 is an irregular singular point otherwise.

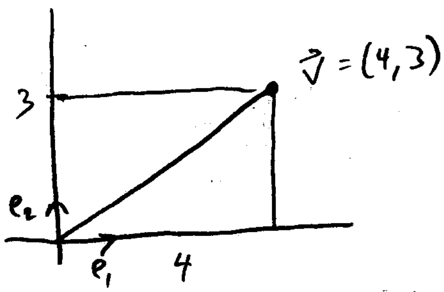
Theorem of Frobenius:

* If x_0 is an ordinary point, then there is a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$
* If x_0 is a regular singular point, then there is a generalized power series solution $y(x) = x^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$.

Moreover, the radius of convergence $R \doteq \min\{R_p, R_Q\}$.

[2]

Let's revisit basic geometry

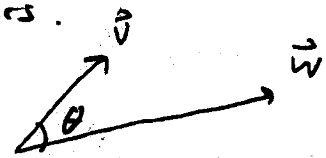


Question: How long is $\vec{v}$ in the x-direction

Ans: 4 (duh).

Better ans: $\vec{v} \cdot \vec{e}_1 = (4,3)\cdot(1,0) = 4.$

* **Big idea:** The dot product allows us to define angles, and hence distances, between vectors.



Fact: $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$

This is actually one way to define angles.

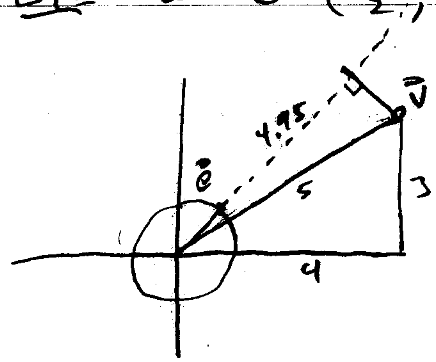
Fact: $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$ Check: $\vec{v} = (4,3)$.

$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{4^2+3^2} = 5$ ✓

* If $\|\vec{e}\| = 1$, then $\vec{v}\cdot\vec{e} =$ "length of $\vec{v}$ in the $\vec{e}$-direction"

$=$ projection of $\vec{v}$ onto $\vec{e}$.

Ex: Let $\vec{e} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\vec{v} = (4,3)$



$\vec{v}\cdot\vec{e} = (4,3)\cdot\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 7\frac{\sqrt{2}}{2} \approx 4.95$

* **Goal:** We want to put a dot product on the space of periodic functions, so we can use these "geometric tools" to analyze them.

First, let's extend the notion of a dot product in $\mathbb{R}^n$, to an arbitrary vector space (we call it an "inner product")

<u>Def:</u> Let X be a vector space. An <u>inner product</u> is a function (denoted, e.g., $\langle u, v\rangle$) such that

(i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ (additive)

(ii) $\langle cu, v\rangle = c\langle u, v\rangle$ (constants pull out)

(iii) $\langle v, w\rangle = \langle w, v\rangle$ (symmetric)

(iv) $\langle v, v\rangle \geq 0$ (non-negative*)

(v) $\langle v, v\rangle = 0$ iff $v = 0$.

<u>Def:</u> If $\langle v, w\rangle = 0$, then v & w are <u>orthogonal</u> (perpendicular)

<u>Example</u>. Consider $\mathbb{R}^3$. Let $\vec{v} = (4,3,2)$.

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an <u>orthonormal basis</u> for $\mathbb{R}^3$, i.e.

* Unit length: $\|e_i\| = 1$, i.e., $\boxed{\langle e_i, e_i\rangle = 1}$
* Orthogonal/(perpendicular): $\boxed{\langle e_i, e_j\rangle = 0}$ if $i \neq j$

$$\vec{v} = (4,3,2) = (\vec{v}\cdot\vec{e}_1, \vec{v}\cdot\vec{e}_2, \vec{v}\cdot\vec{e}_3).$$

* Orthonormal bases are really nice!

We have an inner product for $\mathbb{R}^n$ (the dot product).
It makes $\{\vec{e}_1, \dots, \vec{e}_n\}$ an orthonormal basis.

Now, let $Per_{2\pi}$ = space of 2π-periodic functions.

Define an inner product as follows:

$$\boxed{\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, g(t)\, dt}$$

Fact 1: $\mathcal{B} = \{ \frac{1}{2}, \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \ldots \}$ is a basis for $Per_{2\pi}$

Fact 2: This inner products makes $\mathcal{B}$ an orthonormal basis!

i.e., $\langle \cos nt, \cos mt \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nt \cos mt \, dt = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$.

$\langle \sin nt, \sin mt \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nt \sin mt \, dt = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$

$\langle \cos nt, \sin mt \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nt \sin mt \, dt = 0$.

* Big idea: Since $\mathcal{B}$ is a basis, every 2π-periodic function $\boxed{f(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nt + b_n \sin nt}$

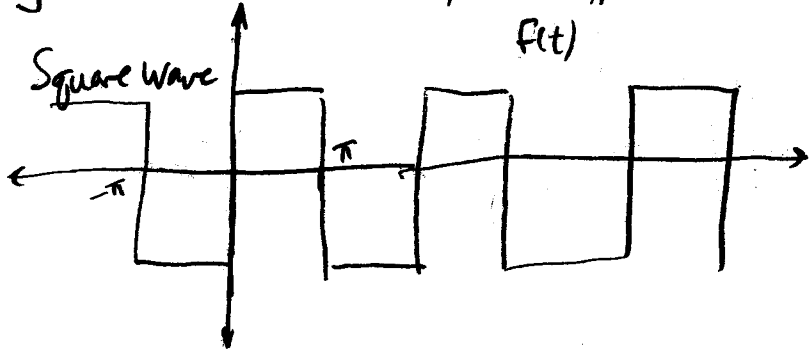
Compare: In $\mathbb{R}^2$, $(4,3)\cdot(1,0) = 4$ "magnitude in x-direction"

In $Per_{2\pi}$, $\langle \vec{v}, \cos 2t \rangle =$ "magnitude in $\cos 2t$-direction"

* The dot/inner product allows us to decompose vectors into components by projection.

So,
$$\boxed{\begin{aligned} a_n &= \langle f(t), \cos nt \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \\ b_n &= \langle f(t), \sin nt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \end{aligned}}$$

Ex 1: Square wave



Find the Fourier series of f(t).

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt = \frac{1}{\pi}\int_{-\pi}^{0} (-1)\, dt + \frac{1}{\pi}\int_0^{\pi} 1\, dt = 0.$$

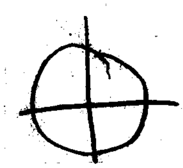
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt = \frac{1}{\pi}\int_{-\pi}^{0} -1 \cos nt\, dt + \frac{1}{\pi}\int_0^{\pi} 1 \cos nt\, dt$$

$$= \frac{-1}{n\pi} \sin nt \Big|_{-\pi}^{0} + \frac{1}{n\pi} \sin nt \Big|_0^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt = \frac{1}{\pi}\int_{-\pi}^{0} -1 \sin nt\, dt + \frac{1}{\pi}\int_0^{\pi} 1 \sin nt\, dt$$

$$= \frac{1}{n\pi} \cos nt \Big|_{-\pi}^{0} - \frac{1}{n\pi} \cos nt \Big|_0^{\pi} = \frac{1}{n\pi}(1 - \cos n\pi) - \frac{1}{\pi}(\cos n\pi - 1)$$

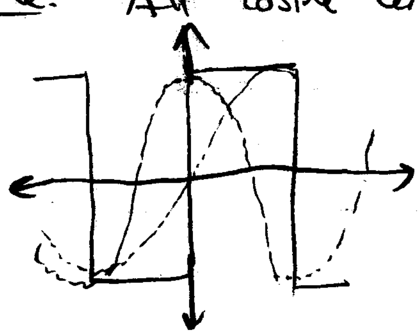
$$= \frac{2}{n\pi}(1 - \cos n\pi).$$ Note: $\cos n\pi = (-1)^n$



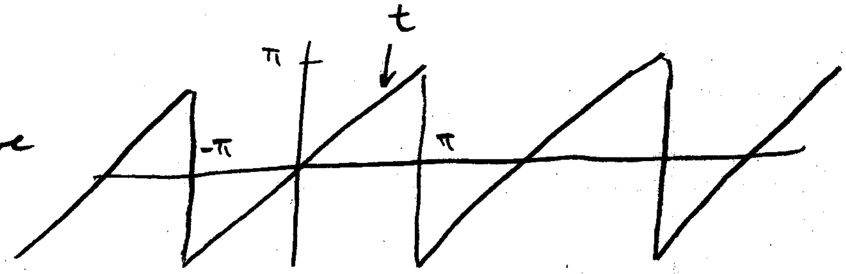
$$\boxed{b_n = \frac{2}{n\pi}(1 - (-1)^n) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}}$$

i.e., $\boxed{f(t) = \frac{4}{\pi} \sin t + \frac{4}{3\pi} \sin 3t + \frac{4}{5\pi} \sin 5t + \frac{4}{7\pi} \sin 7t + \dots}$

Note: All cosine terms, & even sine terms, are zero. (why?)



Ex 2: Sawtooth wave



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = \frac{t^2}{2\pi}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\pi^2 - (-\pi)^2\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \cos nt\,dt = \Big[\text{ let } u = t \quad v = \frac{1}{n}\sin nt$$
$$du = dt \quad dv = \cos nt\,dt$$

$$= \frac{1}{\pi}\left[\underbrace{\frac{1}{n} t \cancel{\sin nt}\Big|_{-\pi}^{\pi}}_{0} - \frac{1}{n}\int_{-\pi}^{\pi} \sin nt\,dt \right]$$

$$= -\frac{1}{n\pi}\int_{-\pi}^{\pi} \sin nt\,dt = \frac{-1}{n^2\pi}\cos nt\Big|_{-\pi}^{\pi} = \frac{1}{n^2\pi}\left[\cos \pi t - \cos(-\pi t)\right]$$
$$= \frac{1}{n^2\pi}\left[\cos \pi t - \cos \pi t\right] = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin nt\,dt \qquad \text{let } u = t \quad v = \frac{-1}{n}\cos nt$$
$$du = dt \quad dv = \sin nt\,dt$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} t \cos nt \Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nt\,dt \right] = \frac{1}{n\pi}\int$$

$$= \frac{1}{\pi n}\left[\left(-\frac{\pi}{n}\cos \pi t\right) - \left(\frac{\pi}{n}\cos \pi t\right) + \underbrace{\frac{1}{n^2}\cancel{\sin nt}\Big|_{-\pi}^{\pi}}_{0} \right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos(\pi t)\right] = \frac{-2}{n}\cos \pi t = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} = \begin{cases} -\frac{2}{n} & n \text{ even} \\ 2/n & n \text{ odd} \end{cases}$$

$$f(t) = 2\sin t - \frac{2}{2}\sin 2t + \frac{2}{3}\sin 3t - \frac{2}{4}\sin 4t + \frac{2}{5}\sin 5t - \frac{2}{6}\sin 6t \pm \ldots$$
$$= 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t \pm \ldots$$

Think: How does this relate to music?