# MTHSC 208, WEEK 12

## Week 11 Summary

- For a vector space $V$, we can define an inner product $\langle \, , \rangle$ (generalized dot product). This allows us to measure, & project vectors.

- For $Per_{2\pi}$, define $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, g(t)\, dt$.

  If $f(t)$ is $2\pi$-periodic, then

  $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

  $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt$$

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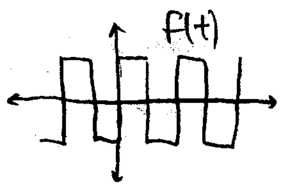
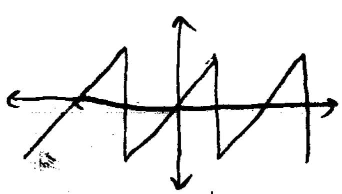
Recall:




$$a_0 = 0, \quad a_n = 0, \quad b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4\pi}{n} & n \text{ odd} \end{cases}$$



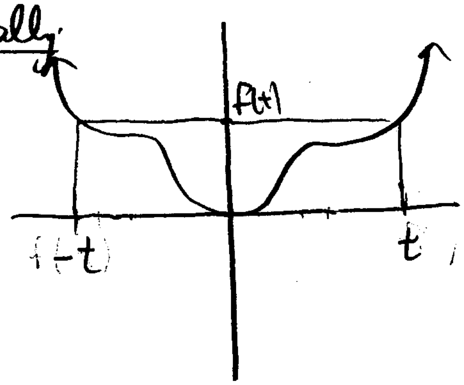
$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{n}(-1)^n$$

Why are many of these coefficients 0 ?

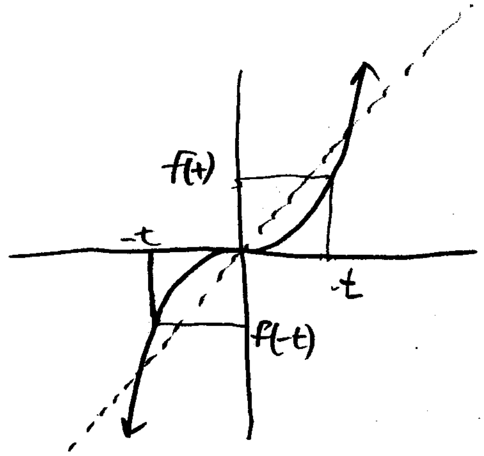
Def: • $f(t)$ is an <u>even function</u> if $f(t) = f(-t)$.

- $f(t)$ is an <u>odd function</u> if $f(t) = -f(-t)$

Graphically:




$f(t)$ even $\Leftrightarrow$ symmetric about y-axis




$f(t)$ odd $\Leftrightarrow$ symmetric about the origin

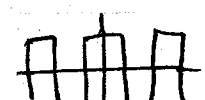
Why we care:

- If $f(t)$ is even: $\int_{-L}^{L} f(t)\,dt = 2\int_0^L f(t)\,dt$
- If $f(t)$ is odd: $\int_{-L}^{L} f(t)\,dt = 0$

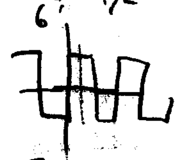
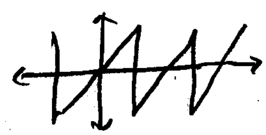
(look at area under curve to see why).

Facts:
- If $f, g$ are even, then $f(t)\,g(t)$ is even.
- If $f, g$ are odd, then $f(t)\,g(t)$ is even
- If $f$ is even, $g$ is odd, then $f(t)\,g(t)$ is odd

Examples:

- Even Functions: $t^2$, $t^4$, $3t^6+t^2-5$, $|t|$, 

  $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} +/- \cdots$

- Odd Functions: $2t$, $8t^3-5t$, , 

  $\sin t = 1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} +/- \cdots$

- Neither: $t^2-3t+2$, $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

If $f(t)$ is even, then $f(t)\cos nt$ is even $\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$

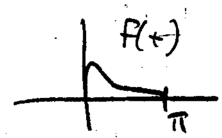
$f(t)\sin nt$ is odd $\Rightarrow$ All $b_n = 0$.

If $f(t)$ is odd, then $f(t)\cos nt$ is odd $\Rightarrow$ All $a_n = 0$.

$f(t)\sin nt$ is even $\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt$

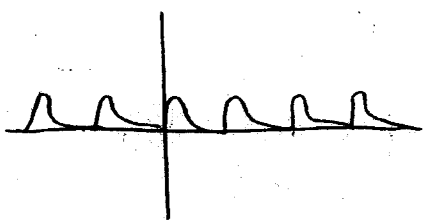
## Fourier sine & cosine series

Idea: Consider some function defined on $[0, \pi]$.

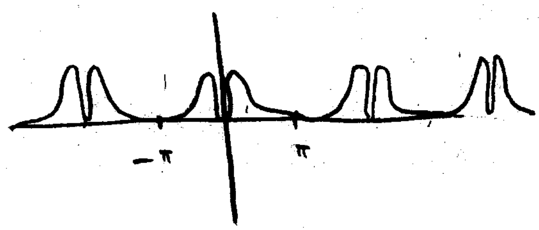



Find the Fourier series of $f(t)$.

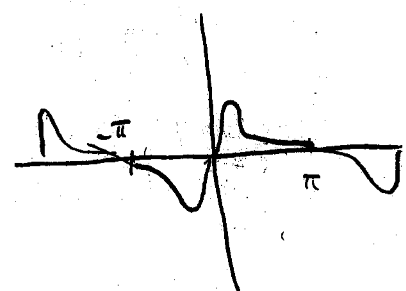
First, we need to make $f(t)$ periodic.



A dumb extension




The even extension




The odd extension

Fourier series of the even extension: $a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$

(called the Fourier Cosine series of $f(t)$) $b_n = 0$

Fourier series of the odd extention: $a_n = 0$

(called the Fourier sine series of $f(t)$) $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt$

Ex: Let $f(t) = t$ on $[0, t]$.

Compute the Fourier sine series and Fourier cosine series

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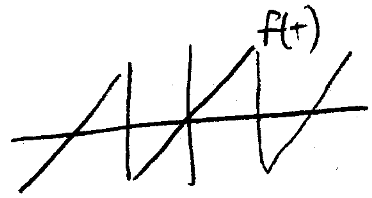
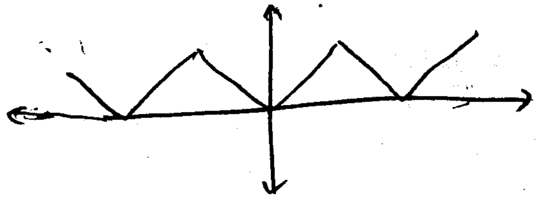
Even extension:




Fourier sine series: $f(t) = \sum_{n=1}^{\infty} b_n \sin nt$, $\quad b_n = \begin{cases} 2/n\pi \\ -2/n\pi \end{cases}$

(we computed this last week).

Odd extension:



Fourier Cosine Series: $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t\, dt = \frac{t^2}{\pi}\Big|_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t \cos nt\, dt = \frac{2}{\pi}\left[ \cancelto{0}{\frac{t}{n}\sin nt\Big|_0^{\pi}} - \int_0^{\pi} \frac{1}{n}\sin nt\, dt \right]$$

let $u = t$ $\quad v = \frac{1}{n}\sin nt$

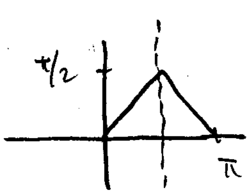
$du = dt$ $\quad dV = \cos nt\, dt$

$$= \frac{2}{n^2\pi}\cos nt\Big|_0^{\pi} = \frac{2}{n^2\pi}\left[\cos n\pi - 1\right]$$

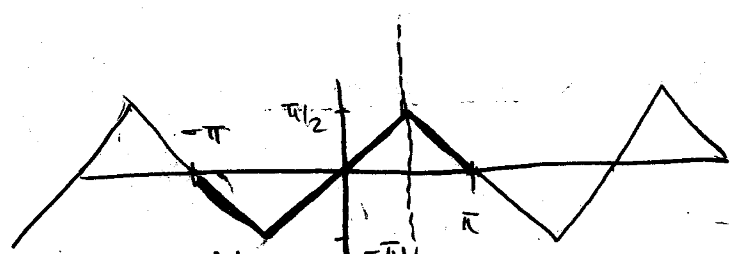
$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right] = \begin{cases} 0 & n \text{ even} \\ -4/n^2\pi & n \text{ odd} \end{cases}$$

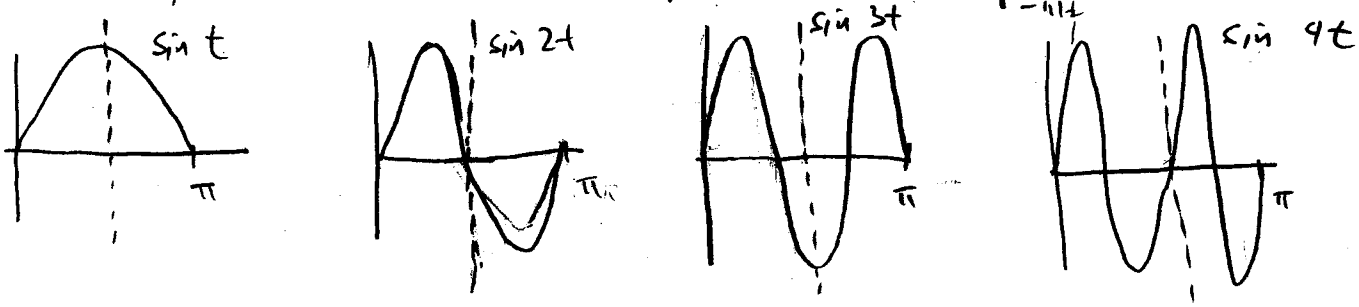
$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\cos t - \frac{4}{9\pi}\cos 3t - \frac{4}{25\pi}\cos 5t - \frac{4}{49\pi}\cos 7t - \cdots$$

Example: $f(t) = \begin{cases} t & 0 \le t \le \pi/2 \\ \pi - t & \pi/2 \le t \le \pi \end{cases}$ Compute the Fourier sine series




Odd extension:



Observe the symmetry about the line $t = \pi/2$.

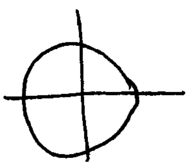
$\sin nt$ is "even about $t = \pi/2$" if $n$ is odd

and is "odd about $t = \pi/2$" if $n$ is even

Since $f(t)$ has even symmetry about $t = \pi/2$, $b_n = 0$ for all even $n$.

and, when $n$ is odd, $f(t)\cos nt$ has even symmetry about $t = \pi/2$.

i.e., $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{4}{\pi}\int_0^{\pi/2} f(t)\sin nt\,dt$



$$= \frac{4}{\pi}\int_0^{\pi/2} t\sin nt\,dt = \frac{4}{\pi}\left[\frac{t}{n}\cos nt\Big|_0^{\pi/2} - \int_0^{\pi/2}\frac{1}{n}\cos nt\,dt\right]$$

$$= \frac{4}{\pi}\left[\frac{\pi}{2n}\cancelto{0}{\cos\left(\frac{n\pi}{2}\right)} - 0 + \frac{1}{n^2}\sin nt\Big|_0^{\pi/2}\right] \quad \text{(since } n \text{ is odd)}$$

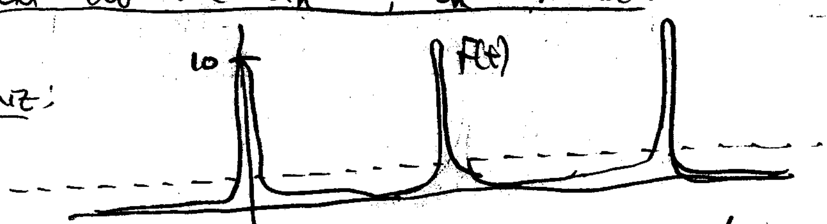
$$= \frac{4}{\pi}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right].$$

Note: $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n = 4k \\ 1 & n = 4k+1 \\ 0 & n = 4k+2 \\ -1 & n = 4k+3 \end{cases}$

$$b_n = \begin{cases} 0 & n = 4k \\ 4/n^2\pi & n = 4k+1 \\ 0 & n = 4k+2 \\ -4/n^2\pi & n = 4k+3 \end{cases}$$

What do the $a_n$'s & $b_n$'s mean?

Quiz:




what is $\frac{a_0}{2}$ (approx)?

Most people say: 10, or 5. (wrong!)

Correct: $\frac{a_0}{2} \approx 1$. Why?

Note: $\frac{a_0}{2}$ is the average value of $f(t)$

Because $f(t) = \frac{a_0}{2} + \underbrace{\sum_{n=1}^{\infty} a_n\cos nt + b_n\sin nt}_{\text{has average value } 0}$

vertical shift by $\frac{a_0}{2}$

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**Quiz:** Express the y-intercept of $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$

Most common answer: $f(0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 0 + b_n \sin 0$

$$\boxed{f(0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n}$$

**Quiz:** Suppose $f(t)$ (extended) has Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$.

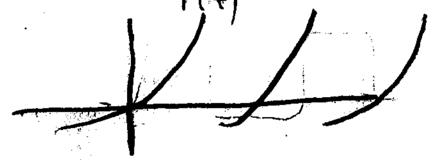
What is the Fourier series of:

(i) $f(t)$ reflected across the y-axis

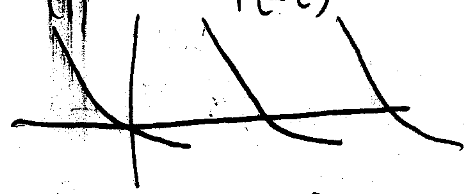
(ii) $f(t)$ reflected across the x-axis

(iii) $f(t)$ reflected across the origin

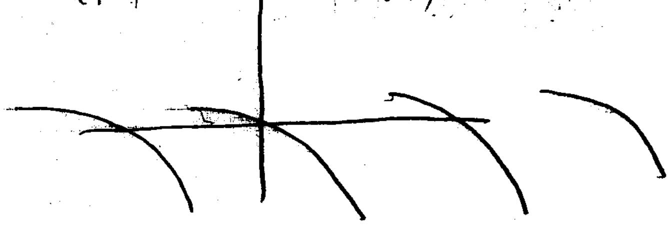
$f(t)$



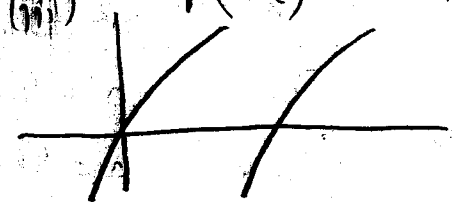
(i) $f(-t)$



(ii) $-f(t)$



(iii) $-f(-t)$



(i) $f(-t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(-nt) + b_n \sin(-nt)$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt - b_n \sin nt$

(ii) $-f(t) = -\frac{a_0}{2} + \sum_{n=1}^{\infty} -a_n \cos nt - b_n \sin nt$

(iii) $-f(-t) = -\frac{a_0}{2} + \sum_{n=1}^{\infty} -a_n \cos nt + b_n \sin nt$

Fact 1: $\mathcal{B}_1 = \{\frac{1}{2}, \cos t, \cos 2t, \dots, \sin t, \sin 2t, \dots\}$ is a basis for $Per_{2\pi}$

and is orthonormal if $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, g(t)\, dt$.

Fact 2: $\mathcal{B}_2 = \{1, e^{-it}, e^{-2it}, e^{-3it}, \dots, e^{it}, e^{2it}, e^{3it}, \dots\}$ is also a basis for $Per_{2\pi}$

and is orthonormal if $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, g(t)\, dt$.

Therefore, if $f(t)$ is $2\pi$-periodic, then

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-int} \qquad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\, dt}$$ e

This is the complex form of the Fourier series of $f(t)$.

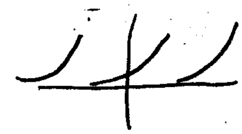
Recall: $\cos nt = \frac{1}{2}(e^{int} + e^{-int})$, $\sin nt = \frac{1}{2i}(e^{int} - e^{-int})$, $e^{int} = \cos nt + i \sin nt$

Therefore, $\boxed{c_n = \frac{a_n - i b_n}{2}, \quad c_{-n} = \frac{a_n + i b_n}{2} \;\Big|\; a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n})}$

Note: $c_0$ is the const. term in complex form of $f(t)$.

$a_0 = 2c_0 \Rightarrow \frac{a_0}{2}$ is the const term in the real form.

Ex: Find the Fourier series of $f(t) = e^t$ on $(-\pi, \pi)$



Method 1: Compute $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^t \cos nt \, dt$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^t \sin nt \, dt$

Method 2: $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t e^{-int} \, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)t} \, dt$

$$= \frac{1}{2\pi} \frac{e^{(1-in)t}}{(1-in)} \Bigg|_{-\pi}^{\pi} = \frac{1}{2\pi(1-int)} \left[ e^{1-in\pi} - e^{-1+in\pi} \right]$$

$$= \frac{(-1)^n}{2(1-in\pi)} \left[ e - \frac{1}{e} \right]$$ since $e^{in\pi} = e^{-in\pi} = (-1)^n$

So $e^t \approx \frac{e - \frac{1}{e}}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-in\pi} e^{in\pi x}$ for $-\pi \le t \le \pi$

Now, compute $a_n$ & $b_n$.

$$a_n = 2 \,\mathrm{Re}\, c_n = \frac{(-1)^n (e - 1/e)}{1 + n^2\pi^2}$$

$$b_n = -2 \,\mathrm{Im}\, c_n = \frac{(-1)^{n+1} n\pi (e - 1/e)}{1 + n^2\pi^2}$$