# MTHSC 208, WEEK 13

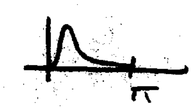
## Week 12 Summary

- Even functions:
  - $f(x) = f(-x)$
  - Symmetric about $y$-axis
  - Fourier series contains only cosine terms
  - $\int_{-L}^{L} f(x)\,dx = 2\int_{-L}^{L} f(x)\,dx$.

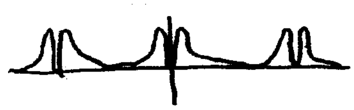
  Odd functions:
  - $f(x) = -f(-x)$
  - Symmetric about origin
  - Fourier series contains only <u>sine terms</u>
  - $\int_{-L}^{L} F(x)\,dx = 0$.

  (even)(even) = even, (odd)(odd) = even, (even)(odd) = odd

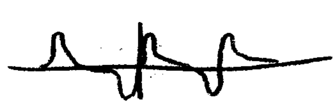
- Fourier cosine & sine series:

  Start w/ a function $f(x)$ on $[0, \pi]$, e.g., 

  - Fourier cosine series is the Fourier series of the <u>even ext</u>.

    e.g.,  $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

  - Fourier sine series is the Fourier series of the <u>odd ext</u>.

    e.g.,  $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

- Real Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$

  Complex Fourier series: $f(x) = C_0 + \sum_{n=1}^{\infty}\left(C_n e^{-inx} + C_{-n}e^{inx}\right)$

  $a_n = C_n + C_{-n}$, $b_n = i(C_n - C_{-n})$, $C_n = \frac{a_n - ib_n}{2}$, $C_{-n} = \frac{a_n + ib_n}{2}$

  Follows from: $e^{ix} = \cos x + i\sin x$, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

  also, $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

**Proposition:** Any function $f(x)$ can be written as:

$$f(x) = \underset{\text{even}}{f_{even}(x)} + \underset{\text{odd}}{f_{odd}(x)}$$

**ex:** $e^{ix} = \cos x + i \sin x = \underbrace{\frac{e^{ix}+e^{-ix}}{2}}_{\text{even}} + \underbrace{\frac{e^{ix}-e^{-ix}}{2i}}_{\text{odd}}$

**Proof:** $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$ $\square$

**Parseval's identity:** If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$,

then $\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 \, dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.

**Proof:** $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right) dx$

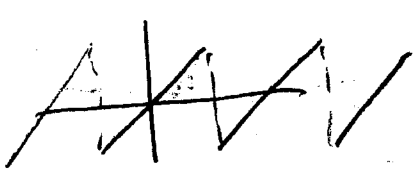
$$= \frac{a_0}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\, dx$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx}_{a_n} + b_n \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx}_{b_n}\right)$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \quad \square$$

**Application:** Compute $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = ?$

Let $f(x) = x$. on $[-\pi, \pi]$



$a_n = 0$ (since $f(x)$ is odd)

$b_n = \frac{2}{n}(-1)^n$ (from last week)

$\Rightarrow b_n^2 = \frac{4}{n^2}$

Parseval $\Rightarrow$ $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^2}{3}$

$$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty}\frac{4}{n^2}.$$

Thus, $4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{3} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$

## Partial differential equations

Let $u(x,t)$ be a 2-variable function. A Partial differential equation (PDE) is an equation involving $u$, $x$, $t$, & the partial derivatives of $u$.

ex: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, or just $u_t = u_{xx}$

ODE have a unifying theory about existance & uniqueness of sol'ns.

PDE have no such unifying theory.

PDE's arise from physical phenomena & modeling.

Heat equation: $\rho(x)\,\sigma(x)\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\,\frac{\partial u}{\partial x}\right)$

where $u(x,t)$ = temperature of a bar at point $x$, time $t$.
$\rho(x)$ = density of bar at point $x$
$\sigma(x)$ = specific heat at point $x$
$\kappa(x)$ = thermal conductivity of the bar

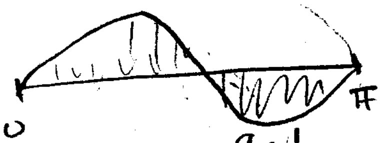
Usually, the bar is homogeneous (i.e., $\rho, \sigma, \kappa$ are const.)

In this case, the heat equation becomes

$$\boxed{\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}}$$

where $k = \frac{\kappa}{\rho\sigma}$

Example: let $u(x,t)$ = temp of a bar of length $\pi$, insulated along the sides, whose ends are kept at zero temp. (boundary conditions)



and, $u(x,t) = \sin 2x$ (initial conditions)

Thus, we have the following initial-value problem:

$$u_t = c^2 u_{xx} \qquad u(0,t)=0, \qquad u(\pi,t)=0, \qquad u(x,0)=\sin 2x$$

Note: this is homogeneous & linear, i.e., if $u_1, u_2$ are solutions, then so is $c_1u_1 + c_2u_2$. (superposition).

Step 1: Find the general solution to $u_t = c^2 u_{xx}$

Assume $u(x,t) = f(x)g(t)$, $u_t = f(x)g'(t)$, $u_{xx} = f''(x)\,g(t)$.
Plug back in & solve for $f$ & $g$.

$$u_t = u_{xx} \rightsquigarrow f(x)g'(t) = f''(x)\,g(t)$$

$$\Rightarrow \frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} = \lambda$$

doesn't depend on $x$ doesn't depend on $t$. $\Rightarrow$ must be a constant!

Now we have 2 ODEs: $\frac{g'(t)}{g(t)} = \lambda$, $\frac{f''(x)}{f(x)} = \lambda$

Solve for $g$: $g' = \lambda g \Rightarrow g(t) = Ae^{\lambda t}$.

Suppose $g(t) \neq 0$. Boundary condition: $u(0,t) = u(\pi,t) = 0$ becomes

$$f(0)g(t) = f(\pi)g(t) = 0 \Rightarrow f(0)=0, \; f(\pi)=0.$$

Solve for F: $f'' = \lambda f \qquad f(0)=0, \qquad f(\pi)=0.$

Case 1: $\lambda = 0$. $\quad f''=0 \Rightarrow f(t) = ax+b. \quad f(0)=0 \Rightarrow a=0$

$f(\pi)=0 \Rightarrow b=0. \Rightarrow f(x)=0.$

Case 2: $\lambda > 0$. $\quad f(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda} x}$

$f(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$f(x) = c_1 \left(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}\right)$

$f(\pi) = c_1 (\underbrace{e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}}_{>0}) = 0 \Rightarrow c_1 = 0.$

$\Rightarrow f(x) = 0.$

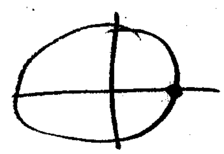
Case 3: $\lambda < 0$. Let $\omega = \sqrt{-\lambda}$, $\quad f'' = -\omega^2 f$

$f(x) = a \cos \omega x + b \sin \omega x.$

$f(0)=0, \qquad f(\pi)=0.$

$f(0) = a = 0 \Rightarrow \quad f(x) = b \sin \omega x$

$f(\pi) = b \sin \omega\pi = 0 \quad \Rightarrow \quad \omega\pi = n\pi$



$\Rightarrow \boxed{\omega = n}.$

Recall, $\omega = \sqrt{-\lambda}$, so $\quad \lambda = -n^2.$

Putting this together, for any integer choice of $n$, we have sol'n:

$u(x,t) = f(x)g(t)$, where $\quad g(t) = Ae^{-n^2 t}$

$f(t) = B \sin nx$

So, $\boxed{u_n(x,t) = A e^{-n^2 t} \sin nx}$ is a sol'n, for any $n$.

By superposition, any linear combination is also a solution.

So the general solution is $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-n^2 t}} \quad (*)$$

Now, let's solve the initial value problem: $u(x,0) = \sin x$

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = \sin 2x \Rightarrow b_2 = 1, \quad b_n = 0 \quad n \neq 2$

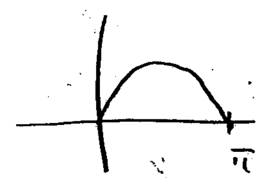
Thus, $(*)$ reduces down to $\boxed{u(x,t) = \sin 2x \, e^{-4t}}$

This is the particular solution to the PDE

$u_t = u_{xx}, \quad u(0,t) = 0, \quad u(\pi,t) = 0, \quad u(x,0) = \sin 2x.$

Question: what if instead we had $u(x,0) = x(\pi - x)$?
(and everything else was the same)

Then we set $\sum_{n=1}^{\infty} b_n \sin nx = x(\pi - x)$




write in terms of sine waves
i.e, find the Fourier sine series

Fourier sine series of $x(\pi - x)$: $\quad b_n = \frac{2}{\pi} \int_{-\pi}^{\pi} x(\pi - x) \sin nx \, dx$

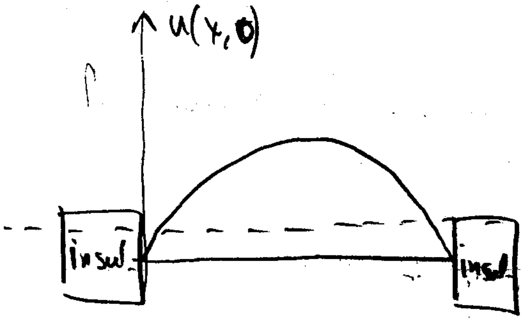
$$= \frac{4}{\pi n^3}\left(1 - (-1)^n\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-n^2 t} = \boxed{\sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx \, e^{-n^2 t}}$$

Note: In both cases, the steady-state solution is $\lim_{t \to \infty} u(x,t) = 0$.
Algebraically, $e^{-n^2 t} \to 0$. Physically, heat dissipates.

Now, consider the same problem, but with different boundary conditions:

$$u_t = u_{xx}, \quad \underbrace{u_x(0,t) = u_x(\pi,t) = 0}_{\text{represented insulated endpoints, through which no heat can escape.}}; \quad u(x,0) = x(\pi - x).$$




--- steady-state solution = average temperature (as we'll see, $\frac{a_0}{2}$ !!)

To solve this, proceed as before, but you'll get

$$f'' = \lambda f$$
$$f'(0) = f'(\pi) = 0 \quad \text{(instead of } f(0) = f(\pi) = 0\text{)}.$$

This has solution $f(x) = a\cos nx + b\sin nx$, $b = 0$

$$\Rightarrow \boxed{f(x) = a\cos nx} \quad \text{(instead of } b \sin nx\text{)}.$$

Thus, the general solution becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} f_n(x)\, g_n(t).$$

$$\Rightarrow \boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, e^{-n^2 t}}$$

To find the particular solution, plug in $t=0$:

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \underbrace{x(\pi - x)}_{\text{express as a Fourier cosine series.}}$$

Recall (from HW): $\quad x(\pi - x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(1-(-1)^n)\cos nx$

Thus, the particular solution is $\boxed{u(x,t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(1-(-1)^n)\cos nx\, e^{-n^2 t}}$

Note: steady state sol'n is $\lim_{t\to\infty} u(x,t) = \frac{\pi^2}{6}$.

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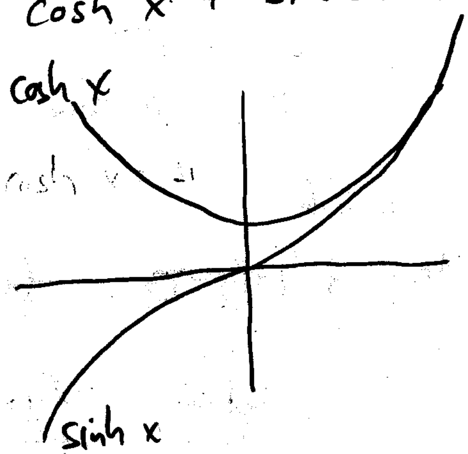
Aside: Recall that for any function $f(x)$,

$$f(x) = f_{even}(x) + f_{odd}(x)$$
$$= \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

ex: $e^{ix} = \frac{e^{ix}+e^{-ix}}{2} + \frac{e^{ix}-e^{-ix}}{2} = \cos x + i \sin x$

$$e^{x} = \frac{e^{x}+e^{-x}}{2} + \frac{e^{x}-e^{-x}}{2} = \cosh x + \sinh x.$$




Note: $\cos 0 = 1$, $\sin 0 = 0$,

$\cosh 0 = 1$, $\sinh 0 = 0$

why we care:

Recall: $y'' = k^2 y$ has general sol'n $y(x) = C_1 e^{kx} + C_2 e^{-kx}$

i.e., $\{e^{kx}, e^{-kx}\}$ is a basis for the sol'n space.

alternatively, it has general sol'n $y(x) = a \cosh kx + b \sinh kx$

i.e., $\{\cosh kx, \sinh kx\}$ is also a basis for the sol'n space.

• While solving the heat equation, we got

$f'' = \lambda f$, $f(0) = 0$, $f(\pi) = 0$.

Case 2: $\lambda > 0 \Rightarrow f(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x} \xrightarrow{\text{annoying math}} C_1 = C_2 = 0$

easier way to do this: $f(x) = a \cosh \omega x + b \sinh \omega x$, $\lambda = \omega^2$.

Now, $f(0) = a = 0 \Rightarrow f(x) = b \sinh \omega x$

$f(\pi) = b \sinh \omega\pi = 0 \Rightarrow b = 0$ (since $\sinh \omega\pi \neq 0$).

Wave equation:

First, consider the following PDE: $\boxed{\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0}$ (*)

Let $f$ be any one-variable function, and set

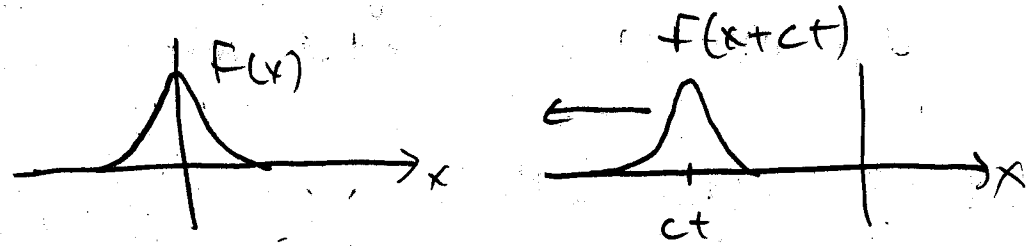
$u(x,t) = f(x+ct)$. $\quad u_x(x,t) = f'(x+ct)$

$\qquad\qquad\qquad\qquad u_t(x,t) = cf'(x+ct)$ (chain rule!)

Note: $u_t - cu_x = cf'(x+ct) - cf'(x+ct) = 0$ ✓

So, $f(x+ct)$ is a solution to (*).

Picture of this:




As $t$ increases, $u(x,t) = f(x+ct)$ is a traveling wave (left).

Next, consider the PDE: $\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0}$ (**)

Let $g$ be any one-variable function and set

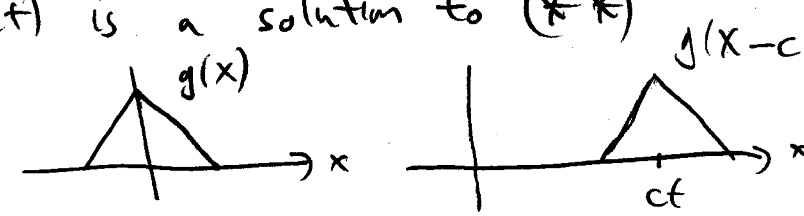
$u(x,t) = g(x-ct)$. $\quad u_x(x,t) = g'(x-ct)$

$\qquad\qquad\qquad\qquad u_t(x,t) = -cg'(x-ct)$ (chain rule!)

Note: $u_t + cu_x = cg'(x-ct) - cg'(x-ct) = 0$ ✓

So, $g(x-ct)$ is a solution to (**)

Picture:




Traveling wave to the right

Now, let $f$ & $g$ be any two functions. Consider the PDE:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = \boxed{\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0} \quad (***)$$

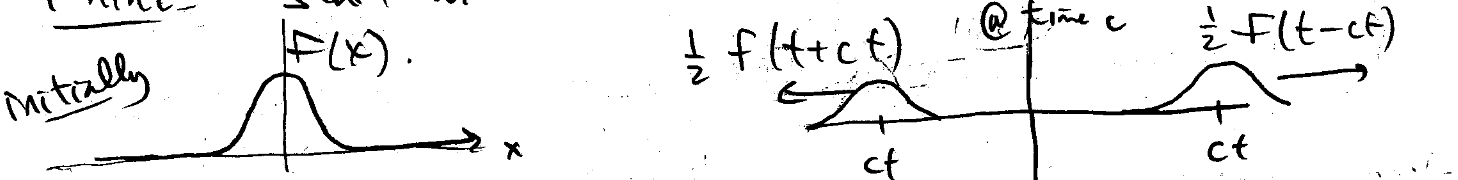
Check that $u(x,t) = f(x+ct) + g(x-ct)$ is a solution.

Consider the following initial conditions with $(***)$:

$u(x,0) = f(x)$. "initial displacement"

$u_t(x,0) = 0$ "initial velocity, i.e, driving force $=0$"

Think: start with a wave in the ocean at time 0.




The solution to this initial value problem is

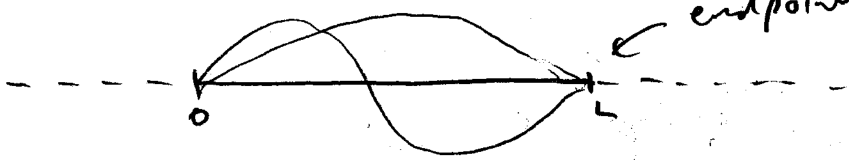
$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct).$$

"half the wave, or energy travels left, half goes right".

Moral: $\boxed{\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2} \text{ is the } \underline{\text{wave equation}}}$

- Suppose we want to model the vibrations of a string / wire of length $L$.

endpoints are fixed



Let $u(x,t)$ be the displacement at point $x$, & time $t$.

Fixed endpoints $\Rightarrow$ $u(0,t)=0$, $u(L,t)=0$.

Must specify initial wave: $u(x,0)=h_1(x)$

and, initial velocity @ $x$: $u_t(x,0)=h_2(x)$
(up/down; not left, right)

Together we get an initial value problem for the wave equation:

$$\boxed{\begin{array}{ll} u_{tt}=c^2 u_{xx} & u(0,t)=0,\ u(L,t)=0 \\ & u(x,0)=h_1(x),\ u_t(x,0)=h_2(x). \end{array}}$$

We'll solve this next week.

(Process is the same as for the heat equation).