MTHSC 208, WEEK 14

Week 13 summary

- Partial differential equations (PDEs): equations involving a function and its partial derivatives

  * Heat equation: $u_t = c^2 u_{xx}$

    Boundary conditions: - (Dirichlet) $u(0,t) = u(L,t) = 0$ (temp of endpts fixed at 0).

    - (Neumann) $u_x(0,t) = u_x(L,t) = 0$ (insulated endpts).

    Initial condition: $u(x,0) = h(x)$: Initial heat distribution of bar.

  * Wave equation: $u_{tt} = c^2 u_{xx}$

    Boundary conditions: $u(0,t) = u(L,t) = 0$ (both endpts fixed)

    Initial conditions: $u(x,0) = h_1(x)$ (initial displacement)

    $u_t(x,0) = h_2(x)$ (initial vertical velocity).

Solving PDEs by separation of variables

Step 1: Assume $u(x,t) = f(x)g(t)$. Plug back in & separate vars.

Step 2: Set resulting equation to const. λ. Get 2 ODEs, one for f, one for g.

Step 3: Solve ODE for g (general solution). Solve ODE for f w/ boundary conditions for f (particular sol'n). Find λ.

Step 4: General solution is $u(x,t) = \sum_{n=0}^{\infty} f_n(x) g_n(t)$.

Step 5: Plug in $t=0$ & use initial conditions to find particular sol'n.

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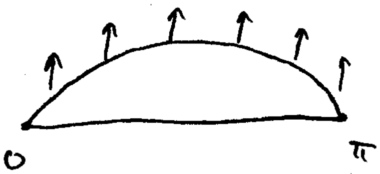
Solving the wave equation



Ex: $u_{tt} = u_{xx}$ $\quad u(0,t) = 0, \quad u(\pi, t) = 0$

$$u(x,0) = x(\pi - x) \qquad u_t(x,0) = 1$$

A solution is a function $u(x,t)$ that describes the vertical displacement at position $x \in [0,\pi]$ and time $t \geq 0$.

Solve using <u>separation of variables</u>: (just like for the heat equation).

Assume $u(x,t) = f(x)\,g(t)$. Plug back in.

$$u_{tt} = f g'', \quad u_{xx} = f'' g \Rightarrow f g'' = f'' g$$

$$\Rightarrow \frac{f''}{f} = \frac{g''}{g} = \lambda \Rightarrow \begin{cases} f'' = \lambda f \\ g'' = \lambda g \end{cases}$$

Moreover, $u(0,t) = f(0)\,g(t) = 0 \Rightarrow f(0) = 0$

$u(\pi,t) = f(\pi)\,g(t) = 0 \Rightarrow f(\pi) = 0$

<u>ODE 1</u>: $f'' = \lambda f$, $\quad f(0) = 0, \quad f(\pi) = 0.$

We've done this: $\boxed{f_n(x) = b_n \sin nx}$ $\qquad \lambda = -n^2$

<u>ODE 2</u>: $g'' = \lambda g \Rightarrow \boxed{g_n(x) = a_n \cos nx + b_n \sin nx}$

Now, for each n, we have a solution

$$u_n(x,t) = f_n(x)\,g_n(t) = (a_n \cos nt + b_n \sin nt) \sin nx$$

By superposition, the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \boxed{\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \sin nx}$$

Finially, let's use our initial conditions.

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}(1-(-1)^n) \sin nx.$$

$$\Rightarrow a_n = \frac{4}{\pi n^3}(1-(-1)^n)$$

(Fourier sine series of $x(\pi-x)$).

$$u_t(x,t) = \sum_{n=1}^{\infty} (-na_n \sin nt + nb_n \cos nt) \sin nx$$

$$u_t(x,0) = \sum_{n=1}^{\infty} nb_n \sin nx = 1 = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n) \sin nx$$

$$\Rightarrow nb_n = \frac{2}{n\pi}(1-(-1)^n)$$

(Fourier sine series of 1).

$$\Rightarrow b_n = \frac{2}{n^2\pi}(1-(-1)^n).$$

Now, our particular solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4}{\pi n^3}(1-(-1)^n) \cos nt + \frac{2}{n\pi}(1-(-1)^n) \sin nt\right] \sin nx}$$

[MIDTERM]

Recall PDEs in 1 (spatial) dimension:

Heat equation: $u_t = c^2 u_{xx}$

Wave equation: $u_{tt} = c^2 u_{xx}$

In 2 dimensions, these PDEs are:

Heat equation: $u_t = c^2(u_{xx} + u_{yy})$

Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$

Let $u(x_1, \dots, x_n)$ be an n-variable function.

The Laplacian of u is $\nabla \cdot \nabla u = \boxed{\nabla^2 u = \sum_{n=1}^{\infty} \frac{\partial^2 u}{\partial x_i^2}}$

In n dimensions, these PDEs are:

Heat equation: $\nabla^2 u = u_t$

Wave equation: $\nabla^2 u = u_{tt}$

(Note: Sometimes the Laplace operator ∇^2 is written Δ).

<u>steady-state solutions</u>: Occur for the heat equation, but not the wave equation (heat diffuses, waves propogate).

Notice that "steady-state" means that $u_t \longrightarrow 0$.
("eventually, the temperature doesn't change w.r.t. time").

Thus, all steady state solutions satisfy

$$\nabla^2 u = 0, \quad \text{i.e.,} \quad \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

<u>Def</u>: A function u is <u>harmonic</u> if $\nabla^2 u = 0$.

<u>Ex</u>: $f(x,y) = x^2 - y^2$. $f_{xx} = 2$, $f_{yy} = -2$, $\nabla^2 f = 0$.

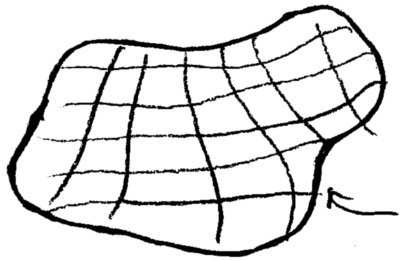
How to visualize harmonic functions:

If $u(x,t)$ is a solution to the heat equation.

Then $\lim_{t\to\infty} \frac{\partial u}{\partial t} = 0$ "temperature will spread out evenly"

* Thus, steady-state solutions to the heat equation are harmonic functions, and are as "flat as possible"

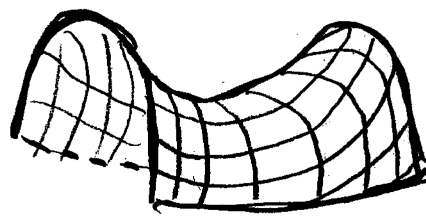
Physical interpretation: stretch out plastic wrap over a bent circular wire, tight as possible.



* The surface is a harmonic function

Fact: If f is harmonic, then for any closed bounded region R, f achieves its min & max values on the boundary, ∂R.

Ex: $f(x, y) = x^2 - y^2$



Picture cutting this surface w/ a "cookie cutter". The max & min points will be on the boundary.

* In other words, there are no local min/maxes.

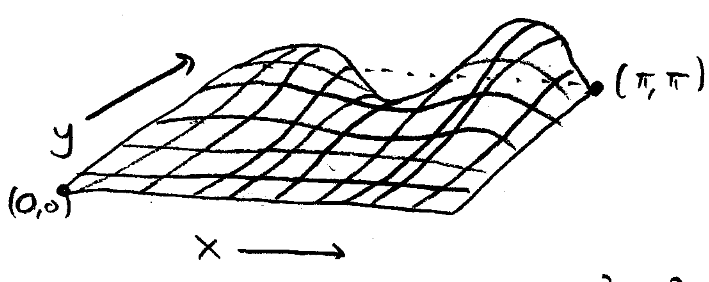
Example: Let $u(x,y)$ be a function, $0 \le x, y \le \pi$, and

$$u_{xx} + u_{yy} = 0,$$

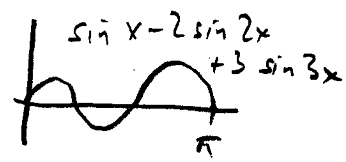
$$u(0,y) = u(\pi, y) = 0$$

$$u(x,0) = 0$$

$$u(x, \pi) = \sin x - 2\sin 2x + 3 \sin 3x$$



Note:



Physical situation: $u(x,y)$ is the steady-state sol'n for the region $[0,\pi] \times [0,\pi]$, where 3 sides are fixed at $0°$, and one at $u(x, \pi)$.

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Step 1: Assume $u(x,y) = X(x)Y(y)$. Plug back in.

$$u_{xx} = X''Y, \quad u_{yy} = XY''.$$

$$u_{xx} + u_{yy} = X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

Also, $u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$

$u(\pi, y) = X(\pi)Y(y) = 0 \Rightarrow X(\pi) = 0$

$u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0.$

Get 2 ODEs: (i) $X'' = \lambda X, \quad X(0) = 0, \quad X(\pi) = 0.$

We know has sol'n $\boxed{X_n(x) = b_n \sin nx}$, $\lambda = -n^2$.

(ii) $Y'' = -\lambda Y \Rightarrow Y'' = n^2 Y, \quad Y(0) = 0.$

Has sol'n $Y(y) = A_n \cosh nx + B_n \sinh nx$

$Y(0) = A_n = 0 \Rightarrow \boxed{Y(y) = B_n \sinh nx}$

Our general solution is thus,

$$u(x,y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y) = \boxed{\sum_{n=1}^{\infty} b_n \sin nx \sinh ny}$$

Finally, we use our last initial condition:

$$u(x,\pi) = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin nx = \sin x - 2\sin 2x + 3\sin 3x$$

$n=1$: $b_1 \sinh \pi \sin x = \sin x \Rightarrow b_1 = \frac{1}{\sinh \pi}$

$n=2$: $b_2 \sinh 2\pi \sin 2x = -2\sin 2x \Rightarrow b_2 = \frac{-2}{\sinh 2\pi}$

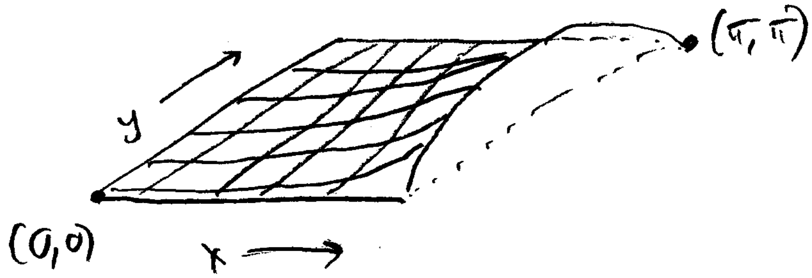
$n=3$: $b_3 \sinh 3\pi \sin 3x = 3\sin 3x \Rightarrow b_3 = \frac{3}{\sinh 3\pi}$

Our particular solution is therefore:

$$\boxed{u(x,y) = \frac{1}{\sinh \pi} \sin x \sinh y - \frac{2}{\sinh 2\pi} \sin 2x \sinh 2y + \frac{3}{\sinh 3\pi} \sin 3x \sinh 3y}$$

<u>Example 2</u>: Suppose we had a similar PDE:

$$u_{xx} + u_{yy} = 0$$
$$u(0,y) = 0$$
$$u(\pi, y) = y(\pi - y)$$
$$u(x,0) = u(x,\pi) = 0$$



Proceed as before, and get

$$Y'' = \lambda Y, \quad Y(0) = Y(\pi) = 0 \Rightarrow Y_n(y) = b_n \sin ny$$

$$X'' = n^2 X, \quad X(0) = 0 \qquad \Rightarrow X_n(x) = B_n \sinh nx.$$

General sol'n: $u(x,y) = \sum_{n=1}^{\infty} b_n \sinh nx \sin ny$

$$u(\pi, y) = y(\pi - y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1 - (-1)^n\right) \sin ny$$

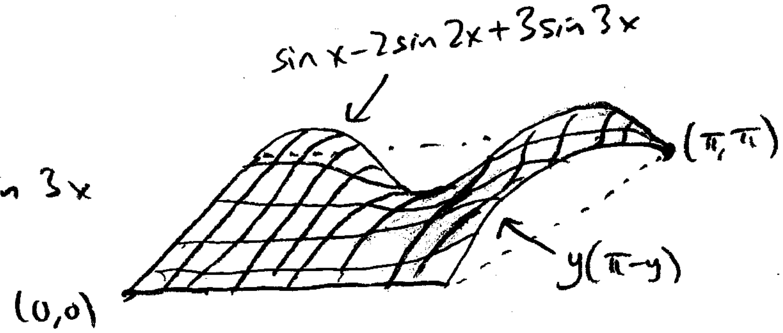
$$= \sum_{n=1}^{\infty} b_n \sinh n\pi \sin ny$$

$$\Rightarrow b_n \sinh n\pi = \frac{4}{\pi n^3}\left(1 - (-1)^n\right) \Rightarrow b_n = \frac{4\left(1-(-1)^n\right)}{\pi n^3 \sinh n\pi}$$

General sol'n: $\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4\left(1-(-1)^n\right)}{\pi n^3 \sinh n\pi} \sinh nx \sin ny}$

Now, consider a superposition of the two previous examples:

$$u_{xx} + u_{yy} = 0$$
$$u(0,y) = 0, \quad u(x,0) = 0$$
$$u(\pi, y) = \sin x - 2\sin 2x + 3\sin 3x$$
$$u(x, \pi) = y(\pi - y)$$



Intuitively, the solution $u(x,y)$ (think steady-state solution of these two PDEs; superimposed boundary conditions) should be the superposition (sum) of the two solutions from the previous examples.

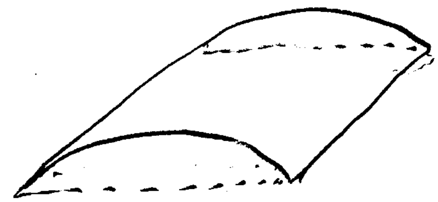
i.e.,

$$u_1(x,y) \quad + \quad u_2(x,y) \quad = \quad u(x,y).$$

Thus, the general solution is

$$\boxed{\begin{aligned} u(x,y) = &\left(\frac{1}{\sinh\pi}\sin x\sinh y + \frac{8}{\pi\sinh\pi}\sinh x\sin y\right) \\ &+\left(\frac{2}{\sinh 2\pi}\sin 2x\sinh 2y\right) \\ &+\left(\frac{3}{\sinh 3\pi}\sin 3x\sinh 3y + \frac{8}{27\pi\sinh 3\pi}\sinh 3x\sin 3y\right) \\ &+\sum_{n=4}^{\infty}\frac{4(1-(-1)^n)}{\pi n^3\sinh n\pi}\sin nx\sinh ny \end{aligned}}$$

Heat equation in 2D:



$u_t = u_{xx} + u_{yy}$, $\quad 0 \le x, y, \le \pi$

$u(0,y,t) = u(\pi,y,t) + u(x,0,t) = u(x,\pi,t) = 0$

$u(x,y,0) = 2\sin x \sin 2y + 3 \sin 4x \sin 5y$.

BC's: Boundary fixed at 0°.

IC: Initial heat distribution.

f^n of pos. $\quad$ f^n of time

Assume sol'n has the form $u(x,y,t) = f(x,y)g(t)$

$u_t = u_{xx} + u_{yy} \rightsquigarrow g'f = g f_{xx} + g f_{yy}$

$$\Rightarrow \frac{g'}{g} = \frac{f_{xx}+f_{yy}}{f} = \frac{\Delta f}{f} = \lambda.$$

Get: $g' = \lambda g \Rightarrow g(t) = Ce^{\lambda t}$

$\Delta f = \lambda f$ "Helmholz equation"

$u(0,y,t) = f(0,y)g(t) = 0 \Rightarrow f(0,y) = 0$

likewise, $f(\pi,y) = f(x,0) = f(x,\pi) = 0$.

Need to solve $f_{xx} + f_{yy} = \lambda f$.

Assume $f(x,y) = X(x)Y(y)$. $\quad f_{xx} = X''Y$, $f_{yy} = XY''$

Plug back in: $\dfrac{X''Y + XY''}{XY} = \dfrac{\lambda XY}{XY} \Rightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} = \lambda.$

$$\Rightarrow \underbrace{\frac{X''}{X}}_{\text{depends only on } x} = \underbrace{\lambda - \frac{Y''}{Y}}_{\text{depends only on } y} = \mu$$

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Get 2 ODEs:

call this ν

$X'' = \mu X, \quad Y'' = (\lambda - \mu) Y \Rightarrow \lambda - \mu = \nu \Rightarrow$

$Y'' = \nu Y. \qquad \Rightarrow \lambda = \nu + \mu$

Recall boundary condition: $f(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$

$f(\pi, y) = 0 \Rightarrow X(\pi) = 0$

$f(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0$

$f(x,\pi) = X(x)Y(\pi) \Rightarrow Y(\pi) = 0.$

$X'' = \mu X, \quad X(0) = X(\pi) = 0 \Rightarrow X_n(x) = b_n \sin nx \quad \mu = -n^2$

$Y'' = \nu Y, \quad Y(0) = Y(\pi) = 0 \Rightarrow Y_m(y) = B_n \sin ny, \quad \nu = -m^2.$

Recall, $\lambda = \nu + \mu = -(n^2 + m^2).$

Thus, for each pair m & n, we have a sol'n

$$\boxed{f_{mn}(x,y) = b_{nm} \sin nx \sin my}$$

$$g_{nm}(t) = Ce^{\lambda t} = \boxed{C_{nm} e^{-(n^2+m^2)t}}$$

The general solution is thus

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin nx \sin my \, e^{-(n^2+m^2)t}}$$