Week 13 Summary

- Partial differential equations (PDEs): equations involving a function and its partial derivatives

- **Heat equation**: \( U_t = c^2 U_{xx} \)
  - Boundary conditions:
    - Dirichlet: \( U(0,t) = u_l(t) = 0 \) (temp of ends fixed at 0).
    - Neumann: \( U_x(0,t) = u_x(l,t) = 0 \) (insulated ends).
  - Initial condition: \( U(X,0) = h(x) \): Initial heat distribution of bar.

- **Wave equation**: \( U_{tt} = c^2 U_{xx} \)
  - Boundary conditions: \( U(0,t) = u_l(t) = 0 \) (both ends fixed).
  - Initial conditions:
    - \( U(X,0) = h_1(x) \) (initial displacement)
    - \( U_t(X,0) = h_2(x) \) (initial vertical velocity).

Solving PDEs by separation of variables:

**Step 1**: Assume \( U(x,t) = f(x)g(t) \). Plug back in & separate vars.

**Step 2**: Set resulting equation to const. \( \lambda \). Get 2 ODEs, one for \( f \), one for \( g \).

**Step 3**: Solve ODE for \( g \) (general solution). Solve ODE for \( f \) w/ boundary conditions for \( f \) (particular soln). Find \( \lambda \).

**Step 4**: General solution is \( U(x,t) = \sum_{n=0}^{\infty} F_n(x)G_n(t) \).

**Step 5**: Plug in \( t=0 \) & use initial conditions to find particular soln.
Solving the wave equation

\[ u_{tt} = u_{xx}, \quad u(0, t) = 0, \quad u(\pi, t) = 0 \]
\[ u(x, 0) = f(x) \quad u_t(x, 0) = 1 \]

A solution is a function \( u(x, t) \) that describes the vertical displacement at position \( x \in [0, \pi] \) and time \( t \geq 0 \).

Solve using separation of variables: (just like for the heat equation).

Assume \( u(x, t) = f(x)g(t) \). Plug back in.

\[ u_{tt} = f''g, \quad u_{xx} = f''g \Rightarrow f''g = f''g \]
\[ \Rightarrow \frac{f''}{f} = \frac{g''}{g} = \lambda \Rightarrow \begin{cases} f'' = \lambda f \\ g'' = \lambda g \end{cases} \]

Moreover,

\[ u(0, t) = f(0)g(t) = 0 \Rightarrow f(0) = 0 \]
\[ u(\pi, t) = f(\pi)g(t) = 0 \Rightarrow f(\pi) = 0 \]

ODE 1: \( f'' = \lambda f \), \( f(0) = 0 \), \( f(\pi) = 0 \).

We've done this:

\[ f_n(x) = \sin(n x) \quad \lambda = -n^2 \]

ODE 2: \( g'' = \lambda g \)

\[ g_n(x) = a_n \cos(n x) + b_n \sin(n x) \]

Now, for each \( n \), we have a solution

\[ u_n(x, t) = f_n(x)g_n(t) = (a_n \cos(nt) + b_n \sin(nt)) \sin(n x) \]

By superposition, the general solution is

\[ u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) \sin(n x) \]
Finally, let's use our initial conditions.

\[ u(x, 0) = \sum_{n=1}^{\infty} a_n \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{n \pi^3} (1 - (-1)^n) \sin nx. \]

\[ \Rightarrow a_n = \frac{4}{n \pi^3} (1 - (-1)^n) \]

(Fourier sine series of \( x(\pi - x) \)).

\[ u_t(x, t) = \sum_{n=1}^{\infty} \left( -n a_n \sin nt + n b_n \cos nt \right) \sin nx \]

\[ u_t(x, 0) = \sum_{n=1}^{\infty} n b_n \sin nx = 1 = \sum_{n=1}^{\infty} \frac{2}{n \pi} (1 - (-1)^n) \sin nx \]

\[ \Rightarrow n b_n = \frac{2}{n \pi} (1 - (-1)^n) \]

(Fourier sine series of 1).

\[ \Rightarrow b_n = \frac{2}{n^2 \pi} (1 - (-1)^n). \]

Now, our particular solution is

\[ u(y, t) = \sum_{n=1}^{\infty} \left[ \frac{4}{n \pi^3} (1 - (-1)^n) \cos nt + \frac{2}{n \pi} (1 - (-1)^n) \sin nt \right] \sin nx \]

[MIDTERM]

Recall PDEs in 1 (spatial) dimension:

Heat equation: \( u_t = C^2 u_{xx} \)

Wave equation: \( u_{tt} = C^2 u_{xx} \)

In 2 dimensions, these PDEs are:

Heat equation: \( u_t = C^2 (u_{xx} + u_{yy}) \)

Wave equation: \( u_{tt} = C^2 (u_{xx} + u_{yy}) \)

Let \( u(x_1, \ldots, x_n) \) be an \( n \)-variable function.

The Laplacian of \( u \) is \( \nabla \cdot \nabla u = \nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \)
In n dimensions, these PDEs are:

Heat equation: \( \nabla^2 u = u_t \)

Wave equation: \( \nabla^2 u = u_{tt} \)

(Note: Sometimes, the Laplace operator \( \nabla^2 \) is written \( \Delta \).

Steady-state solutions: Occur for the heat equation, but not the wave equation (heat diffuses, waves propagate).

Notice that "steady-state" means that \( u_t \to 0 \).
("eventually, the temperature doesn't change w.r.t. time").

Thus, all steady state solutions satisfy

\[ \nabla^2 u = 0, \text{ i.e., } \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0. \]

Def: A function \( u \) is harmonic if \( \nabla^2 u = 0 \).

Ex: \( f(x,y) = x^2 - y^2 \). \( f_{xx} = 2 \), \( f_{yy} = -2 \), \( \nabla^2 f = 0 \).

How to visualize harmonic functions:

If \( u(x,t) \) is a solution to the heat equation,

Then \( \lim_{t\to\infty} \frac{\partial u}{\partial t} = 0 \) "temperature will spread out evenly".

Thus, steady-state solutions to the heat equation are harmonic functions, and are as "flat as possible".
Physical interpretation: stretch out plastic wrap over a bent circular wire, tight as possible. The surface is a harmonic function.

Fact: If $f$ is harmonic, then for any closed bounded region $R$, $f$ achieves its min & max values on the boundary, $\partial R$.

Example: let $u(x,y)$ be a function, $0 \leq x, y \leq \pi$, and

$U_{xx} + U_{yy} = 0, \quad u(0,y) = u(\pi,y) = 0, \quad u(x,0) = 0, \quad u(x,\pi) = \sin x - 2 \sin 2x + 3 \sin 3x$

Physical situation: $u(x,y)$ is the steady-state solution for the region $[0,\pi] \times [0,\pi]$, where 3 sides are fixed at 0°, and one at $u(x,\pi)$. 

Picture cutting this surface with a "cookie-cutter." The max & min points will be on the boundary.

In other words, there are no local min/maxes.
Step 1: Assume \( U(x, y) = X(x) Y(y) \). Plug back in.

\[
\begin{align*}
X'' + Y'' = 0 & \implies \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \\
X'' + \lambda X = 0 & \implies X(0) = 0 \quad X(\pi) = 0 \\
Y'' - \lambda Y = 0 & \implies Y(0) = 0
\end{align*}
\]

Also,

\[
\begin{align*}
X(0) Y(y) = 0 & \implies X(0) = 0 \\
X(\pi) Y(y) = 0 & \implies X(\pi) = 0 \\
X(x) Y(0) = 0 & \implies Y(0) = 0.
\end{align*}
\]

Get 2 ODEs: (i) \( X'' = \lambda X \), \( X(0) = 0 \), \( X(\pi) = 0 \).

We know has soln \( X_n(x) = b_n \sin n x \), \( \lambda = -n^2 \).

(ii) \( Y'' = -\lambda Y \) \( \implies Y'' = n^2 Y \), \( Y(0) = 0 \).

Has soln \( Y_n(y) = A_n \cosh n y + B_n \sinh n y \)

\[
\begin{align*}
X_n(x) = b_n \sinh n x & \implies Y_n(y) = b_n \sinh n y
\end{align*}
\]

Our general solution is thus

\[
U(x, y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y) = \sum_{n=1}^{\infty} b_n \sinh n x \sinh n y
\]

Finally, we use our last initial condition.

\[
U(x, \pi) = \sum_{n=1}^{\infty} (b_n \sinh n \pi) \sin n x = \sin x - 2 \sin 2x + 3 \sin 3x
\]

\[
\begin{align*}
N = 1: \quad b_1 \sinh \pi \sin x = \sin x & \implies b_1 = \frac{1}{\sinh \pi} \\
N = 2: \quad b_2 \sinh 2\pi \sin 2x = -2 \sin 2x & \implies b_2 = \frac{-2}{\sinh 2\pi} \\
N = 3: \quad b_3 \sinh 3\pi \sin 3x = 3 \sin 3x & \implies b_3 = \frac{3}{\sinh 3\pi}
\end{align*}
\]

Our particular solution is therefore:

\[
U(x, y) = \frac{1}{\sinh \pi} \sin x \sinh y - \frac{2}{\sinh 2\pi} \sin 2x \sinh 2y + \frac{3}{\sinh 3\pi} \sin 3x \sinh 3y
\]
Example 2: Suppose we had a similar PDE:

\[ u_{xx} + u_{yy} = 0 \]

\[ u(0, y) = 0 \]

\[ u(\pi, y) = y(\pi - y) \]

\[ u(x, 0) = u(x, \pi) = 0 \]

Proceed as before, and get

\[ Y'' = \lambda Y, \quad Y(0) = Y(\pi) = 0 \Rightarrow Y_n(y) = b_n \sin ny \]

\[ X'' = \mu^2 X, \quad X(0) = 0 \Rightarrow X_n(x) = B_n \sin \mu x \]

General soln: \[ u(x, y) = \sum_{n=1}^{\infty} b_n \sinh \mu x \sin ny \]

\[ u(\pi, y) = y(\pi - y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3} (1 - (-1)^n) \sin ny \]

\[ = \sum_{n=1}^{\infty} b_n \sinh n\pi \sin ny \]

\[ \Rightarrow b_n \sinh n\pi = \frac{4}{\pi n^3} (1 - (-1)^n) \Rightarrow b_n = \frac{4(1 - (-1)^n)}{\pi n^3 \sinh n\pi} \]

General soln: \[ u(x, y) = \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{\pi n^3 \sinh n\pi} \sinh nx \sin ny \]

Now, consider a superposition of the two previous examples:

\[ u_{xx} + u_{yy} = 0 \]

\[ u(0, y) = 0, \quad u(x, 0) = 0 \]

\[ u(\pi, y) = \sin x - 2 \sin 2x + 3 \sin 3x \]

\[ u(x, \pi) = y(\pi - y) \]
Intuitively, the solution \( u(x, y) \) (think steady-state solution of these two PDEs; superimposed boundary conditions) should be the superposition (sum) of the two solutions from the previous example.

\[
\begin{align*}
U_1(x, y) + U_2(x, y) &= U(x, y) \\
\end{align*}
\]

Thus, the general solution is

\[
\begin{align*}
u(x, y) &= \left( \frac{-1}{\sinh \pi} \sin x \sinh y + \frac{8}{\pi \sinh \pi} \sinh x \sin y \right) \\
&\quad - \left( \frac{2}{\sinh 2\pi} \sin 2x \sinh 2y \right) \\
&\quad + \left( \frac{3}{\sinh 3\pi} \sin 3x \sinh 3y + \frac{8}{27\pi \sinh 3\pi} \sinh 3x \sin 3y \right) \\
&\quad + \sum_{\eta=4}^{\infty} \frac{4(1-(-1)^{\eta})}{\pi \eta^3 \sinh \eta \pi} \sin \eta x \sinh \eta y \\
\end{align*}
\]
Heat equation in 2D:

\[ u_t = u_{xx} + u_{yy}, \quad 0 \leq x, y, \leq \pi \]

\[ u(0, y, t) = u(\pi, y, t) + u(x, 0, t) = u(x, \pi, t) = 0 \]

\[ u(x, y, 0) = 2 \sin x \sin 2y + 3 \sin x \sin 5y. \]

**BC's:** Boundary fixed at 0°.

**IC:** Initial heat distribution.

Assume solution has the form \( u(x, y, t) = f(x, y)g(t) \).

\[ u_t = u_{xx} + u_{yy} \quad \Rightarrow \quad g\frac{d^2f}{dx^2} + g\frac{d^2f}{dy^2} \]

\[ \Rightarrow \quad \frac{g'}{g} = \frac{f_{xx} + f_{yy}}{f} = \Delta f = \lambda. \]

Act. \[ g' = \lambda g \quad \Rightarrow \quad g(t) = Ce^{\lambda t}. \]

\[ \Delta f = \lambda f \quad \text{"Helmholtz equation"} \]

\[ u(0, y, t) = f(0, y)g(t) = 0 \quad \Rightarrow \quad f(0, y) = 0 \]

Likewise, \( f(\pi, y) = f(x, 0) = f(x, \pi) = 0 \).

Need to solve \( f_{xx} + f_{yy} = \lambda f \).

Assume \( f(x, y) = X(x)Y(y) \).

\[ f_{xx} = X''Y, \quad f_{yy} = XY'' \]

Plug back in: \[ X''Y + XY'' = \lambda XY \quad \Rightarrow \quad \frac{X''}{X} - \frac{Y''}{Y} = \lambda. \]

\[ \Rightarrow \quad \frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu \]

depends only on \( x \) \quad \text{depends only on } \mu \text{ only on } x, y
\textbf{Get 2 ODEs:}

\[ X'' = \mu X, \quad Y'' = (\lambda - \mu) Y \quad \Rightarrow \quad \lambda - \mu = \nu \quad \Rightarrow \quad \lambda = \nu + \mu \]

Recall boundary condition:

\[ f(0, y) = X(0) Y(y) = 0 \Rightarrow X(0) = 0 \]
\[ f(\pi, y) = 0 \Rightarrow X(\pi) = 0 \]
\[ f(x, 0) = X(x) Y(0) = 0 \Rightarrow Y(0) = 0 \]
\[ f(x, \pi) = X(x) Y(\pi) \Rightarrow Y(\pi) = 0. \]

\[ X'' = \mu X, \quad X(0) = X(\pi) = 0 \Rightarrow X_n(x) = b_n \sin nx \quad \nu = -n^2 \]
\[ Y'' = \nu Y, \quad Y(0) = Y(\pi) = 0 \Rightarrow Y_n(y) = B_n \sin ny, \quad \nu = -m^2. \]

Recall, \( \lambda = \nu + \mu = -(n^2 + m^2). \)

Thus, for each pair \( m \text{ and } n, \) we have a solution:

\[ f_{nm}(x, y) = b_{nm} \sin nx \sin my \]
\[ g_{nm}(t) = Ce^{\lambda t} = C_{nm} e^{-(n^2 + m^2)t}. \]

The general solution is thus:

\[ u(x, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin nx \sin my e^{-(n^2 + m^2)t} \]