MTHSC 208, WEEK 15

Week 14 summary

- Solving the wave equation: $u_{tt} = c^2 u_{xx}$. Assume $u(x,t) = f(x)g(t)$.

  Eigenvalue equation (if $c=1$): $\frac{f''}{f} = \frac{g''}{g} = \lambda$

  $f(x) = a \cos \omega x + b \sin nx$, $g(t) = A \cos \omega t + B \sin \omega t$.

- Laplace operator: $\nabla^2 f = \Delta f = \sum \frac{\partial^2 f}{\partial x_i^2}$

  A function f is harmonic if $\Delta f = 0$.

  Harmonic functions: * Are as "flat as possible" (think: plastic wrap)

  * Have no local mins/maxs.
  * Model steady-state temperature distribution

- Higher dimensional PDEs:

  * Heat equation: $\Delta u = u_t$
  * Wave equation: $\Delta u = u_{tt}$
  * Laplace's equation: $\Delta u = 0$

- Solving Laplace's equation (in 2D): $u_{xx} + u_{yy} = 0$

  Assume $u(x,y) = X(x)\,Y(y)$.

  Eigenvalue equation: $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \Rightarrow$ $X_n(x) = a_n \cos nx + b_n \sin nx$
  $Y_n(y) = A_n \cosh ny + B_n \sinh ny$.

  Finally, use boundary conditions to find a_n, b_n, A_n, B_n.

  Solution is the "plastic wrap surface" stretched over the boundary.

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- <u>Solving the heat equation</u>: (In 2D): $u_{xx}+u_{yy}=u_t$

Assume $u(x,t) = F(x,y)\, g(t)$.

Eigenvalue equation: $\frac{\Delta F}{F} = \frac{g'}{g} = \lambda \Rightarrow$ $F_{xx}+F_{yy}=\lambda$ (Helmholtz eqn)
$g(t) = Ce^{\lambda t}$.

To solve Helmholtz: Assume $F(x,y) = X(x)\,Y(y)$.

Eigenvalue equation: $\frac{X''}{X} + \frac{Y''}{Y} = \lambda \Rightarrow \frac{X''}{X} = \mu, \quad \frac{Y''}{Y} = \nu$.

$\mu = -m^2, \quad \nu = -n^2, \quad \lambda = \mu+\nu = -(m^2+n^2)$

$X_m(x) = a_m \cos mx + b_m \sin mx, \quad Y_n(y) = A_n \cos ny + B_n \sin ny$.

$g_{mn}(t) = C_{mn} e^{-(m^2+n^2)t}, \quad F_{mn}(x,y) = X_m(x)\, Y_n(y)$.

$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_{mn}(x,y)\, g_{mn}(t)$. use initial condition.

Steady state: $u_t \approx 0 \Rightarrow \Delta u = 0$.

This week: Solving the 2D wave equation, & special topics.

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(1) <u>Laplace transforms</u>: $\mathcal{L}\{y(t)\}(s) = \int_0^\infty y(t) e^{-st}\,dt = Y(s)$

Turned time-derivatives into multiplication by s: $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

$\mathcal{L}\{y''(t)\} = s^2 Y(s) - s\,y(0) - y'(0)$.

We can also take $\mathcal{L}\{u(x,t)\}(x,s) = \int_0^\infty u(x,t) e^{-st}\,dt = U(x,s)$.

similarly, $\mathcal{L}\{u_t(x,t)\} = sU(x,s) - u(x,0)$.

$\mathcal{L}\{u_{tt}(x,t)\} = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$.

$\mathcal{L}\{u_x(x,t)\} = U_x(x,s)$.

$\mathcal{L}$ turns: ODEs into algebraic equations.

PDEs into ODEs. (if e.g., $u(x,t)$).

$$\begin{tikzcd} \text{ODE} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{algebraic equation} \arrow[d, "\text{solve}"] \\ \text{ODE solution} & \text{solution} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}
\qquad
\begin{tikzcd} \text{PDE} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{ODE} \arrow[d, "\text{solve}"] \\ \text{PDE solution} & \text{ODE solution} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

(2) Fourier transforms: $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$

Turns spatial-derivatives into multiplication by $2\pi i \xi$.

$$\widehat{f'(x)} = 2\pi i \xi\, \hat{f}(\xi) \quad \text{or more generally,}$$

$$\widehat{f^{(n)}(x)} = (2\pi i \xi)^n \hat{f}(\xi).$$

Similarly, $\widehat{u_x(x,t)} = (2\pi i \xi)\, U(\xi, t)$, etc

$$\widehat{u_t(x,t)} = U_t(\xi, t)$$

Thus, Fourier transforms turn PDEs into ODEs.

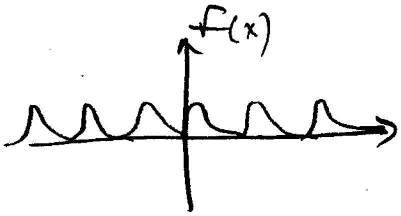
$$\begin{tikzcd} \text{PDE} \arrow[r, "\mathcal{F}"] \arrow[d, dashed] & \text{ODE} \arrow[d, "\text{solve}"] \\ \text{PDE solution} & \text{ODE solution} \arrow[l, "\mathcal{F}^{-1}"'] \end{tikzcd}$$

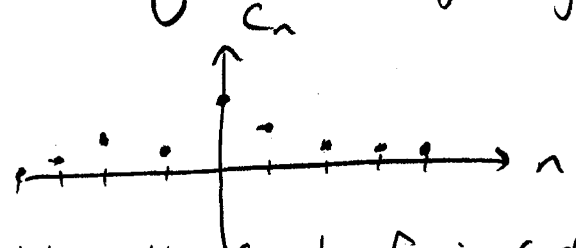
And, the inverse Fourier transform is easy: it's (almost) the same!)

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi.$$

A neat way to think of Fourier (& Laplace) transforms.

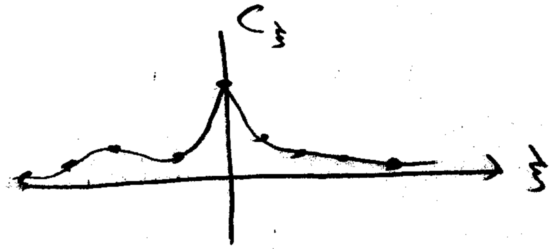
Let $f(x)$ be 2π-periodic. We can plot f as a function of x (space) OR as a function of n (frequency)





plot of the complex fourier coefficients.

Question: what would a function look like if we allowed a continuum of frequencies?



It need not even be periodic.

Start with $f(x)$ $\longrightarrow$ $\sum_{n=-\infty}^{\infty} c_n e^{-inx}$ Function of n (frequency)

↑ function of x

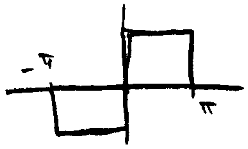
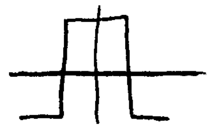
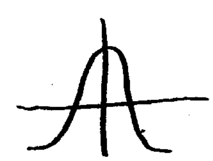
$\longrightarrow$ $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, d\xi$

Function of ξ (frequency).

(3) Wavelets:

Big idea: Many periodic functions aren't smooth, so representing them as sines & cosines isn't natural.

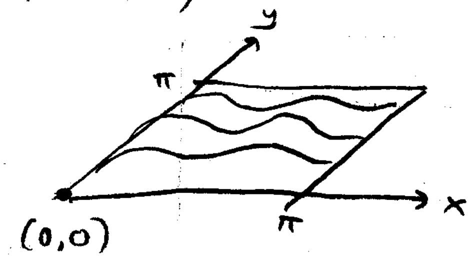
Solution: Use a different basis for $Per_{2\pi}$.

e.g.,  &  instead of  and 

These are called the Haar wavelets.

The 2D Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$

Let $u(x,y,t)$ = displacement of a square membrane of length π.



Suppose the boundary is held fixed.

This is modeled by the following PDE (say $c=1$)

$$u_{tt} = u_{xx} + u_{yy}$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0 \quad \text{(BC's)}.$$

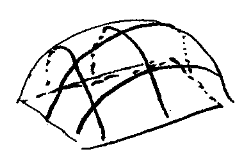
$$u(x,y,0) = p(x)\,q(y), \quad u_t(x,y,0) = 0 \quad \text{(IC's)}$$

Let's solve this if $p(x) = x(\pi - x)$, $q(y) = y(\pi - y)$.

Recall that on $[0,\pi]$, $p(x) = \sum_{m=1}^{\infty} \frac{4}{\pi m^3}\left(1-(-1)^m\right) \sin mx$

$$q(y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin ny.$$

This initial wave is like a paraboloid:



Step 1: Assume $u(x,y,t) = f(x,y)\,g(t)$.

Plug back in: $fg'' = f_{xx}g + f_{yy}g \Rightarrow \frac{f_{xx}+f_{yy}}{f} = \frac{g''}{g} = \lambda$

Step 2: Get boundary conditions for f & g:

$u(x,0,t) = f(x,0)\,g(t) = 0 \Rightarrow f(x,0) = 0$

Similarly, $f(x,\pi) = f(0,y) = f(\pi,y) = 0$.

Also, $u_t(x,y,0) = f(x,y)\,g'(0) = 0 \Rightarrow g'(0) = 0$.

Step 3: Solve eigenvalue equation: (for f)

$f_{xx} + f_{yy} = \lambda f$. Assume $f(x,y) = X(x)\, Y(y)$

Plug back in: $\dfrac{X''Y + XY''}{XY} = \dfrac{\lambda XY}{XY}$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \lambda \Rightarrow \underbrace{\frac{X''}{X}}_{\text{independent of } y} = \underbrace{\frac{-Y''}{Y} + \lambda}_{\text{independent of } x} = \mu \quad \Rightarrow \text{Const.}$$

① $\dfrac{X''}{X} = \mu$, $X(0) = X(\pi) = 0 \Rightarrow \boxed{\mu = -m^2 \text{ and } X_n(x) = b_n \sin mx}$.

② $\dfrac{Y''}{Y} = \lambda - \mu := \nu$ $\quad Y(0) = Y(\pi) = 0 \Rightarrow \boxed{\nu = -n^2 \text{ and } Y_n(y) = B_n \sin ny}$

Thus, $\boxed{f_{mn}(x,y) = b_{mn} \sin mx \sin ny}$, $\lambda = \mu + \nu$
$\Rightarrow \boxed{\lambda = -(m^2+n^2)}$

Step 4: Solve eigenvalue equation (for g).

$g'' = \lambda g \Rightarrow g'' = -(m^2+n^2) g$

$\Rightarrow g_{mn}(t) = A_{mn} \cos(\omega_{mn} t) + B_{mn} \sin(\omega_{mn} t)$

where $\omega_{mn} = \sqrt{-\lambda} = \sqrt{m^2+n^2}$

use $g'(0) = 0$: $\quad g'(t) = -\omega_{mn} A_{mn} \sin(\omega_{mn} t) + B_{mn} \omega_{mn} \cos(\omega_{mn} t)$

$g'(0) = B_{mn} \omega_{mn} \Rightarrow B_{mn} = 0$.

Therefore, $\boxed{g_{mn}(t) = A_{mn} \cos\left(\sqrt{m^2+n^2}\, t\right)}$

Step 4: Write down general solution:

$$u(x,y,t) = \sum_{m,n \geq 0} F_{mn}(x,y)\, g_{mn}(t)$$

$$= \boxed{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \left(\sin mx \sin ny\right) \cos\left(\sqrt{m^2+n^2}\, t\right)}$$

Step 5: Use initial conditions to find particular sol'n:

$$u(x,y,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin mx \sin ny$$

$$= p(x)\, q(y) = \left[x(\pi - x)\right]\left[y(\pi - y)\right]$$

$$= \sum_{m=1}^{\infty} \frac{4}{\pi m^3}\left(1-(-1)^m\right) \sin mx \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin ny$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16}{\pi^2 m^3 n^3} \left(1-(-1)^m\right)\left(1-(-1)^n\right) \sin mx \sin ny$$

Thus,

$$\boxed{b_{mn} = \frac{16\,\overbrace{\left(1-(-1)^m\right)}^{0 \text{ or } 2}\,\overbrace{\left(1-(-1)^n\right)}^{0 \text{ or } 2}}{\pi^2 m^3 n^3}} = \begin{cases} \dfrac{64}{\pi^2 m^3 n^3} & m \,\&\, n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$