MTHSC 208, HW 15

(1) Use the ratio test to find the radius of convergence of the following power series:

a. \[\sum_{n=0}^{\infty} (-1)^n x^n,\]
b. \[\sum_{n=0}^{\infty} \frac{1}{n+1} (x-\pi)^n,\]
c. \[\sum_{n=0}^{\infty} \frac{3}{n+1} (x-2)^n,\]
d. \[\sum_{n=0}^{\infty} \frac{1}{2n} (x-\pi)^n,\]
e. \[\sum_{n=0}^{\infty} (5x-10)^n,\]
f. \[\sum_{n=0}^{\infty} \frac{1}{n!} (3x-6)^n.\]

(2) Use the comparison test to find an estimate for the radius of convergence of each of the following power series:

a. \[\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n},\]
b. \[\sum_{n=0}^{\infty} (-1)^n x^{2n},\]
c. \[\sum_{n=1}^{\infty} \frac{1}{2n} (x-4)^{2n},\]
d. \[\sum_{n=0}^{\infty} \frac{1}{2n^2} (x-\pi)^{2n}.\]

(3) Use the comparison test and the ratio test to find the radius of convergence of the power series

\[\sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.\]

(4) The differential equation

\[(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + p(p+1)y = 0,\]

where \(p\) is a constant, is known as Legendre’s equation. We will solve it by first assuming that the solution is a power series centered at 0.

(a) Find the recursion formula for \(a_{n+2}\) in terms of \(a_n\).
(b) Use the recursion formula to determine \(a_n\) in terms of \(a_0\) and \(a_1\), for \(2 \leq n \leq 9\).
(c) Find a nonzero polynomial solution to this differential equation, in the case where \(p = 3\).
(d) Find a basis for the space of solutions to the differential equation.

\[(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 12y = 0,\]

(5) The differential equation

\[(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + p^2 y = 0,\]

where \(p\) is a constant, is known as Chebyshev’s equation. It can be rewritten in the form

\[\frac{d^2y}{dx^2} - P(x) \frac{dy}{dx} + Q(x)y = 0, \quad \text{where} \quad P(X) = -\frac{x}{1-x^2}, \quad Q(x) = \frac{p^2}{1-x^2}.\]

(a) If \(P(x)\) and \(Q(x)\) are represented as a power series about \(x_0 = 0\), what is the radius of convergence of these power series?
(b) Assuming a power series centered at 0, find the recursion formula for \(a_{n+2}\) in terms of \(a_n\).
(c) Use the recursion formula to determine \(a_n\) in terms of \(a_0\) and \(a_1\), for \(2 \leq n \leq 9\).
(d) In the special case where \(p = 3\), find a nonzero polynomial solution to this differential equation.