MTHSC 208, HW 16

(1) For each of the following sets, determine if it is a vector space. If it is, give a basis. If it isn't, explain why not.
  (a) The set of points in $\mathbb{R}^3$ with $x = 0$.
  (b) The set of points in $\mathbb{R}^2$ with $x = y$.
  (c) The set of points in $\mathbb{R}^3$ with $x = y$.
  (d) The set of points in $\mathbb{R}^3$ with $z \geq 0$.
  (e) The set of unit vectors in $\mathbb{R}^2$.
  (f) The set of polynomials of degree n.
  (g) The set of polynomials of degree at most n.
  (h) The set of polynomials of degree at most n, with even constant term.
  (i) The set of polynomials of degree at most n, with odd constant term.

(2) Let X be the set of polynomials of degree at most 4. Give three different examples of a basis for X, and one non-example.

(3) Let X be a vector space over $\mathbb{C}$ (i.e., the contants are complex numbers, instead of just real numbers). If $\{v_1, v_2\}$ is a basis of X, then by definition, every vector v can be written uniquely as $v = C_1v_1 + C_2v_2$.
  (a) Is the set $\{v_1 + v_2, 3v_1 - 2v_2\}$ also a basis of X?
  (b) Is the set $\{\frac{1}{2}v_1 + \frac{1}{2}v_2, \frac{1}{2i}v_1 - \frac{1}{2i}v_2\}$ a basis of X?
  (c) Consider the ODE $y'' + 4y = 0$. If we assume that $y(t) = e^{rt}$, then we get that $r = \pm 2i$. Therefore, the general solution is $y(t) = C_1e^{2it} + C_2e^{-2it}$, i.e., $\{e^{2it}, e^{-2it}\}$ is a basis for the solution space. Use (b), and Euler's equation ($e^{i\theta} = \cos\theta + i\sin\theta$) to find a basis for the solution space involving sines and cosines, and write the general solution using sines and cosines.

(4) We will find the general solution of *Airy's equation*: $y'' + xy = 0$.
  (a) Assume the solution is a power series. Find the recurrence relation of the coefficients of the power series. *Hint: When shifting the indices, one way is to let $m = n - 3$, then factor out x^{n+1} and find a_{n+3} in terms of a_n. Alternatively, you can find a_{n+2} in terms of a_{n-1}.)*
  (b) Show that $a_2 = 0$. *Hint: the two series for y'' and xy don't "start" at the same power of x, but for any solution, each term must be zero.*
  (c) Find the particular solution when $y(0) = 1$, $y'(0) = 0$, as well as the particular solution when $y(0) = 0$, $y'(0) = 1$.
  (d) Find the radii of convergence of the two series from (c).

(5) For each of the following ODEs, determine whether $x = 0$ is an ordinary or singular point. If it is singular, determine whether it is regular or not. (Remember, first write each ODE in the form $y'' + P(x)y' + Q(x)y = 0$.)
  (a) $y'' + xy' + (1 - x^2)y = 0$
  (b) $y'' + (1/x)y'' + (1 - (1/x^2))y = 0$.
  (c) $x^2y'' + 2xy + (\cos x)y = 0$.
  (d) $x^3y'' + 2xy' + (\cos x)y = 0$.