MTHSC 208, HW 19

If $f(t)$ is a 2π-periodic function, then $f(t)$ can be written *uniquely* as

(1) $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \;,$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \;, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad (n \geq 1)\;, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \quad (n \geq 1)\;.$$

The sum in (1) is the *Fourier series* of $f(t)$. Alternatively, $f(t)$ can be written as

(2) $$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-int}\,dt \;,$$

where

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

The sum in (2) is the *complex form of the Fourier series* of $f(t)$.

Simplify your final results, by using $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$.

Exercises:

(1) Determine which of the following functions are even, which are odd, and which are neither even nor odd:
   a. $f(t) = t^3 + 3t$.
   b. $f(t) = t^2 + |t|$.
   c. $f(t) = e^t$.
   d. $f(t) = \frac{1}{2}(e^t + e^{-t})$.
   e. $f(t) = \frac{1}{2}(e^t - e^{-t})$.

(2) (Section 12.1 #29): The Fourier series of an odd function consists only of sine-terms. What additional symmetry conditions on f will imply that the sine coefficients with even indices will be zero? Give an example of a function satisfying this additional condition.

(Section 12.1, #31): Suppose that f is a function defined on $\mathbb{R}$ (not necessarily periodic). Show that there is an odd function f_{odd} and an even function f_{even} such that $f(x) = f_{\text{odd}} + f_{\text{even}}$.

(3) Consider the function defined on the interval $[0, \pi]$:
$$f(t) = \begin{cases} t & \text{for } 0 \leq t < \pi/2, \\ \pi - t, & \text{for } \pi/2 \leq t \leq \pi. \end{cases}$$
   a. Sketch the even extension of this function and find its Fourier cosine series.
   b. Sketch the odd extension of this function and find its Fourier sine series.

(4) Consider the function defined on the interval $[0, \pi]$:
$$f(t) = t(\pi - t).$$
   a. Sketch the even extension of this function and find its Fourier cosine series.
   b. Sketch the odd extension of this function and find its Fourier sine series.

(5) a. Find the complex Fourier coefficients of the function

$$f(t) = t^2 \qquad \text{for } -\pi < t \leq \pi,$$

extended to be periodic of period 2π.

b. Find the real form of the Fourier series. *Hint: Use* $a_n = c_n + c_{-n}$, *and* $b_n = i(c_n - c_{-n})$.