MTHSC 208, HW 20

If $f(t)$ is a 2π-periodic function, then $f(t)$ can be written *uniquely* as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \ , \tag{1}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \ , \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad (n \geq 1) \ , \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \quad (n \geq 1) \ .$$

The sum in (1) is the *Fourier series* of $f(t)$. Alternatively, $f(t)$ can be written as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-int}\,dt \ , \tag{2}$$

where

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

The sum in (2) is the *complex form of the Fourier series* of $f(t)$. The coefficients are related by:

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n}) \ , \tag{3}$$

$$c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2} \ . \tag{4}$$

Simplify your final results, by using $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$.

Exercises:

1. Compute the complex Fourier series for the function defined on the interval $[-\pi, \pi]$:
$$f(x) = \begin{cases} -1, & -\pi \leq x < 0, \\ 1, & 0 \leq x \leq \pi. \end{cases}$$
Use (3) to find the real Fourier series (the a_n's and b_n's).

2. Find the real and complex Fourier series for the function defined on the interval $[-\pi, \pi]$:
$$f(x) = \begin{cases} 0, & -\pi \leq x < 0, \\ 1, & 0 \leq x \leq \pi. \end{cases}$$
Only compute one of these directly (your choice), and use (3) or (4) to compute the other.

3. Compute the complex Fourier series for the function $f(x) = \pi - x$ defined on the interval $[-\pi, \pi]$. Use (3) to find the real version of the Fourier series.

4. Prove Parseval's identity:
$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \ .$$

5. Use Parseval's identity, and the Fourier series of the function $f(x) = x^2$ on $[-\pi, \pi]$, to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.