MTHSC 208, HW 21

The partial differential equation
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$
is the heat equation, where $u(x,t)$ is the temperature of a length-L bar at point x and time t. To solve this PDE, we assume the solution has the form $u(x,t) = f(x)g(t)$, plug this back in, and solve for f and g. Usually there is an *initial condition*, $u(x,0) = h(x)$, and two *boundary conditions*.

The *Dirichlet boundary conditions*, $u(0,t) = u(L,t) = 0$, model the scenerio that the endpoints are fixed at 0 degrees. Alternatively, the *Neumann boundary conditions*, $u_x(0,t) = u_x(L,t) = 0$, model the scenerio that heat cannot escape from the endpoints (i.e., they're insulated).

Together, the heat equation, along with an initial condition, and two boundary conditions, form an *initial value problem*, to which there is one unique solution.

(1) We will find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:
$$\frac{1}{2t+1}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = \sin x + 3\sin 2x - 5\sin 7x.$$
(a) Assume that $u(x,t) = f(x)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ). Solve for $g(t)$, $f(x)$, and λ.

(b) Using your solution to (a), find the general solution to the PDE
$$\frac{1}{2t+1}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x^2}$$
subject to the Dirichlet boundary conditions: $u(0,t) = u(\pi,t) = 0$.

(c) Solve the initial value problem, i.e., find the particular solution $u(x,t)$ that satisfies $u(x,0) = \sin x + 3\sin 2x - 5\sin 7x$.

(d) What is the steady-state solution, i.e., $\lim_{t\to\infty} u(x,t)$?

(2) Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:
$$\frac{1}{2t+1}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = x(\pi - x).$$
Note: The general solution will be exactly the same as in the previous problem. All you need to do again is Part (c) and (d) for this new initial condition, $u(x,0) = x(\pi - x)$.

(3) We will find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u_x(0,t) = u_x(\pi,t) = 0, \qquad u(x,0) = 3\cos x + 7\cos 2x\ .$$
(a) Assume that $u(x,t) = f(x)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ). Solve for $g(t)$, $f(x)$, and λ.

(b) Using your solution to (a), find the general solution to the PDE
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x^2}$$
subject to the Neumann boundary conditions: $u_x(0,t) = u_x(\pi,t) = 0$.

(c) Finally, solve the initial value problem, i.e., find the particular solution $u(x,t)$ that satisfies $u(x,0) = 4 + 3\cos 2x - 8\cos 7x$.

(d) What is the steady-state solution?