MTHSC 208, HW 22

(1) We will solve for $u(\theta, t)$, the two-variable function that describes the temperature of a circular wire, and satisfies the heat equation with *periodic boundary conditions*:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \qquad u(\theta + 2\pi, t) = u(\theta, t), \qquad u(\theta, 0) = 2 + 4\sin 3\theta - \cos 5\theta.$$

(a) Once again, assume that $u(\theta, t) = f(\theta)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ). *Note: The notation is easier when you write the derivatives in the numerator, e.g., $g'(t)/g(t) = \lambda$, rather than $g(t)/g'(t) = \lambda$.*

(b) Find the general solution to the ODEs for $g(t)$ and $f(\theta)$.

(c) Use the periodic boundary conditions for $u(\theta, t)$ to derive similar periodic boundary conditions for $f(\theta)$. Solve for ω (or equivalently, λ) in the general solution *Note: You won't be able to conclude that $a = 0$ or $b = 0$ – so unlike before, they'll both stick around.*

(d) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(\theta, t) = f_n(\theta)g_n(t)$.

(e) Find the particular solution to the initial value problem by using the initial condition.

(f) What is the steady-state solution?

(2) Find the function $u(x, t)$ defined for $0 \le x \le \pi$ and $t \ge 0$ which satisfies the following initial value problem of the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0, t) = u(\pi, t) = 0,$$
$$u(x, 0) = 8\sin x + 11\sin 2x + 15\sin 4x, \qquad \frac{\partial u}{\partial t}(x, 0) = 0.$$

(a) Once again, assume that $u(x, t) = f(x)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ).

(b) Find the general solutions to the ODEs for $g(t)$ and $f(x)$.

(c) Use the boundary conditions for $u(x, t)$ to derive similar boundary conditions for $f(x)$. Solve for $f(x)$ and ω (or equivalently, λ).

(d) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(\theta, t) = f_n(\theta)g_n(t)$.

(e) Find the particular solution to the initial value problem by using the initial conditions.

(3) In this problem, we will find the function $u(x, t)$ defined for $0 \le x \le \pi$ and $t \ge 0$ which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0, t) = u(\pi, t) = 0,$$
$$u(x, 0) = x(\pi - x), \qquad \frac{\partial u}{\partial t}(x, 0) = 0.$$

Steps (a)–(d) are the same as in the previous problem, and need not be repeated. Instead, Repeat part (e) with this new initial condition. What physical situation does this model? Give a physical interpretation for both boundary conditions, and both of the initial conditions, and sketch this scenerio at time $t = 0$.

(4) In this problem, we will find the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies different initial conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = x(\pi - x).$$

(a) Asume that $u(x,t) = f(x)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ).
(b) Find the general solutions to the ODEs for $g(t)$ and $f(x)$.
(c) Use the boundary conditions for $u(x,t)$ to derive similar boundary conditions for $f(x)$. Solve for $f(x)$ and ω (or equivalently, λ).
(d) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(\theta,t) = f_n(\theta)g_n(t)$.
(e) Find the particular solution to the initial value problem by using the initial conditions.
(f) Give a physical interpretation for both boundary conditions, and both of the initial conditions. How does this differ from the PDE in the previous problem?