MTHSC 208, HW 23

(1) Which of the following functions are harmonic?
  (a) $f(x) = 10 - 3x$.
  (b) $f(x, y) = x^2 + y^2$.
  (c) $f(x, y) = x^2 - y^2$.
  (d) $f(x, y) = e^x \cos y$.
  (e) $f(x, y) = x^3 - 3xy^2$.

(2) (a) Solve the following Dirichlet problem for Laplace's equation in a square region: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = u(\pi, y) = 0,$$
$$u(x, 0) = 0, \quad u(x, \pi) = \sin x - 2\sin 2x + 3\sin 3x$$

  (b) Solve the following Dirichlet problem for Laplace's equation in the same square region: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = 0, \quad u(\pi, y) = \sin 2y + 3\sin 4y,$$
$$u(x, 0) = u(x, \pi) = 0$$

  (c) By adding the solutions to parts a and b togeter, find the solution to the Dirichlet problem: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = 0, \quad u(\pi, y) = \sin 2y + 3\sin 4y,$$
$$u(x, 0) = 0, \quad u(x, \pi) = \sin x - 2\sin 2x + 3\sin 3x$$

  (d) Sketch the solutions to (a), (b), and (c). *Hint: it is enough to sketch the boundaries (using a calculator or computer) – and then use the fact that the solutions are harmonic functions.*

(3) Solve the following initial value problem for the heat equation in a square region: Find $u(x, y, t)$, where $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$ such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0$$
$$u(x, y, 0) = 2\sin x \sin y + 5\sin 2x \sin y$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = \lambda f(x, y), \qquad f(x, 0) = f(x, \pi) = f(0, y) = f(\pi, y) = 0,$$

are of the form

$$\lambda = -(m^2 + n^2), \qquad f(x, y) = b_{mn} \sin mx \sin ny,$$

for $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$, where b_{mn} is a constant.

(4) Solve the following initial value problem for the heat equation in a square region: Find $u(x,y,t)$, where $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$ such that

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0 \\
&u(x,y,0) = 2(\sin x) y(\pi - y).
\end{aligned}$$

Sketch the initial heat distribution over this region. What is the steady-state solution?

(5) Solve the following initial value problem for a vibrating square membrane: Find $u(x,y,t)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\
&u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0 \\
&u(x,y,0) = p(x)q(y), \qquad \frac{\partial u}{\partial t}(x,y,0) = 0.
\end{aligned}$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \le x \le \pi/2, \\ \pi - x, & \text{for } \pi/2 \le x \le \pi, \end{cases} , \qquad p(y) = \begin{cases} y, & \text{for } 0 \le y \le \pi/2, \\ \pi - y, & \text{for } \pi/2 \le y \le \pi. \end{cases}$$

Sketch the initial displacement of the square membrane. What is the long-term behavior of $u(x,y,t)$?