(1) Which of the following functions are harmonic?
   (a) \( f(x) = 10 - 3x. \)
   (b) \( f(x, y) = x^2 + y^2. \)
   (c) \( f(x, y) = x^2 - y^2. \)
   (d) \( f(x, y) = e^x \cos y. \)
   (e) \( f(x, y) = x^3 - 3xy^2. \)

(2) (a) Solve the following Dirichlet problem for Laplace’s equation in a square region: Find \( u(x, y), \) \( 0 \leq x \leq \pi, \) \( 0 \leq y \leq \pi \) such that
   \[
   \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(0, y) = u(\pi, y) = 0, \\
   u(x, 0) = 0, \quad u(x, \pi) = \sin x - 2\sin 2x + 3\sin 3x
   \]

(b) Solve the following Dirichlet problem for Laplace’s equation in the same square region: Find \( u(x, y), \) \( 0 \leq x \leq \pi, \) \( 0 \leq y \leq \pi \) such that
   \[
   \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(0, y) = 0, \quad u(\pi, y) = \sin 2y + 3\sin 4y, \\
   u(x, 0) = u(x, \pi) = 0
   \]

(c) By adding the solutions to parts a and b together, find the solution to the Dirichlet problem: Find \( u(x, y), \) \( 0 \leq x \leq \pi, \) \( 0 \leq y \leq \pi \) such that
   \[
   \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(0, y) = 0, \quad u(\pi, y) = \sin 2y + 3\sin 4y, \\
   u(x, 0) = 0, \quad u(x, \pi) = \sin x - 2\sin 2x + 3\sin 3x
   \]

(d) Sketch the solutions to (a), (b), and (c). \textit{Hint:} it is enough to sketch the boundaries (using a calculator or computer) – and then use the fact that the solutions are harmonic functions.

(3) Solve the following initial value problem for the heat equation in a square region: Find \( u(x, y, t), \) \( 0 \leq x \leq \pi, \) \( 0 \leq y \leq \pi \) and \( t \geq 0 \) such that
   \[
   \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
   u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
   u(x, y, 0) = 2\sin x\sin y + 5\sin 2x\sin y
   \]

You may assume that the nontrivial solutions to the eigenvalue problem
   \[
   \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = \lambda f(x, y), \quad f(x, 0) = f(x, \pi) = f(0, y) = f(\pi, y) = 0,
   \]
   are of the form
   \[
   \lambda = -(m^2 + n^2), \quad f(x, y) = b_{mn} \sin mx \sin ny,
   \]
   for \( m = 1, 2, 3, \ldots \) and \( n = 1, 2, 3, \ldots \), where \( b_{mn} \) is a constant.
(4) Solve the following initial value problem for the heat equation in a square region: Find \( u(x, y, t) \), where \( 0 \leq x \leq \pi, \ 0 \leq y \leq \pi \) and \( t \geq 0 \) such that

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},
\]

\[
u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0
\]

\[
u(x, y, 0) = 2(\sin x)y(\pi - y).
\]

Sketch the initial heat distribution over this region. What is the steady-state solution?

(5) Solve the following initial value problem for a vibrating square membrane: Find \( u(x, y, t) \), \( 0 \leq x \leq \pi, \ 0 \leq y \leq \pi \) such that

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},
\]

\[
u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0
\]

\[
u(x, y, 0) = p(x)q(y), \quad \frac{\partial u}{\partial t}(x, y, 0) = 0.
\]

where

\[
p(x) = \begin{cases} x, & \text{for } 0 \leq x \leq \pi/2, \\ \pi - x, & \text{for } \pi/2 \leq x \leq \pi, \end{cases}
\]

\[
p(y) = \begin{cases} y, & \text{for } 0 \leq y \leq \pi/2, \\ \pi - y, & \text{for } \pi/2 \leq y \leq \pi. \end{cases}
\]

Sketch the initial displacement of the square membrane. What is the long-term behavior of \( u(x, y, t) \)?