MTHSC 208 (Differential Equations)
Dr. Matthew Macauley
HW 3
Due Friday September 5th, 2009

(1) A parachutist of mass 60 kg free-falls from an airplane at an altitude of 5000 meters. He is subjected to an air resistance force proportional to his speed. Assume that the constant of proportionality is $r = 10$ kg/sec.
  (a) Find and solve the differential equation govering the altitude of the parachuter at time t seconds after the start of his free-fall.
  (b) Assuming he does not deploy his parachute, find his limiting velocity and how much time will elapse before he hits the ground.

(2) In our model of air resistance, the resistance force has depended only on the velocity. However, for an object that drops a considerable distance, such as the parachutist in the previous exercise, there is a dependence on the altitude as well. It is reasonable to assume that the resistance force is proportional to air pressure, as well as to the velocity. Furthermore, to a first-order approximation, the air pressure varies exponentially with the altitude (i.e., it is proportional to e^{-ax}, where a is a constant and x is the altitude). Propose and justify (*but do not solve!*) a differential equation model for the velocity of a falling object subject to such a resistance force.

(3) Use the integrating factor method to find the general solution of the following differential equations. Solve the following ODEs using the integrating factor method.
  (a) $2y' - 3y = 5$
  (b) $y' + 2ty = 5t$
  (c) $ty' = 4y + t^4$

(4) Use the varation of parameters method to find the general solution of the following differential equation. Then find the particular solution satisfying the given initial condition.
  (a) $y' - 3y = 4, \quad y(0) = 2$
  (b) $y' + y = e^t, \quad y(0) = 1$
  (c) $y' + 2xy = 2x^3, \quad y(0) = -1.$

(5) Consider Newton's law of cooling, but suppose that the ambient temperature varies sinusoidally with time, as in

$$T' = -k(T - A\sin\omega t)\,.$$

  (a) Solve the homogeneous equation, $T_h' = -kT_h$.
  (b) The ODE above is not autonomous, so finding a particular solution T_p is a bit more difficult. However, it doesn't hurt to guess. As a first guess, substitute $T_p = C\cos\omega t + D\sin\omega t$ into the equation $T' + kT = kA\sin\omega t$ and show that

$$-\omega C + kD = kA \qquad \text{and} \qquad kC + \omega D = 0\,.$$

  (c) Give a qualitative description of what this particular solution represents [Hint: What would happen if initially, the object is the same as the ambient temperature?]
  (d) Solve the simultaneous equations in part (b), and determine the general solution to this ODE.

(6) A murder victim is discovered at midnight at the temperature of the body is recorded at 31°C, and it was discovered that the proportionality constant in Newton's law was $k = \ln(5/4) \approx 0.223$. Assume that at midnight the surrounding air temperature is 0°C, and is falling at a constant rate of 1°C per hour. At what time did the victim die? Remember that the normal body temperature is 37°C. [Hint: Letting $t = 0$ represent midnight will simplify your calculations.]

(7) Suppose that the temperature T inside a mountain cabin behaves according to Newton's law of cooling, as in

$$\frac{dT}{dt} = -\frac{1}{2}(T - A)\,,$$

where t is measured in hours and the ambient temperature A outside the cabin varies sinusoidally with a period of 24 hours. At 6am, the ambient temperature outside is at a minimum of 40°, and at 6pm, the ambient temperature is at a maximum of 80°.

(a) Adjust the differential equation above to model the sinusoidal nature of the ambient temperature.

(b) Suppose that at midnight the temperature inside the cabin is 50°. Solve the resulting initial value problem. [Hint: Letting $t = 0$ represent midnight will simplify your calculations.]

(c) Sketch the graph of the temperature inside the cabin. On the same coordinate system, superimpose the plot of the ambient temperature outside the cabin. Comment on the appearance of the plot.