MTHSC 208 (Differential Equations)
Dr. Matthew Macauley
HW 4
Due Tuesday September 8th, 2009

(1) Solve the differential equation $y' = 2y + 4$ four different ways:
   (a) Finding a constant solution, and writing $y(t) = y_h(t) + y_p(t)$.
   (b) Integrating factor
   (c) Variation of parameters
   (d) Separation of variables

(2) A tank contains 100 gal of pure water. A salt solution with concentration 3 lb/gal enters the tank at a rate of 2 gal/min. Solution drains from the tank at a rate of 2 gal/min. *Without doing any math*, determine the eventual concentration of the salt solution in the tank (i.e., the steady-state solution).

(3) A tank contains 100 gal of pure water. At time zero, a sugar-water solution containing 0.2 lb of sugar per gal enters the tank at a rate of 3 gal per minute. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 gal per minute. Assume that the solution in the tank is kept perfectly mixed at all time.
   (a) What will be the sugar content in the tank after 20 minutes?
   (b) How long will it take the sugar content in the tank to reach 15 lb?
   (c) What will be the eventual sugar content in the tank?

(4) A tank initially contains 50 gal of sugar water having a concentration of 2 lb of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.
   (a) How much sugar is in the tank after 10 minutes?
   (b) How long will it take the sugar content in the tank to dip below 20 lbs?
   (c) What will be the eventual sugar content in the tank?

(5) A tank contains 500 gal of a salt-water solution containing 0.05 lb of salt per gallon of water. Pure water is poured into the tank and a drain at the bottom of the tank is adjusted so as to keep the volume of solution in the tank constant. At what rate (gal/min) should the water be poured into the tank to lower the salt concentration to 0.01 lb/gal of water in under one hour (i.e., $t = 60$ minutes)?

(6) The *logistic equation* is the differential equation

$$y'(t) = ky(t)\left(1 - \frac{y(t)}{M}\right),$$

and it has solution $y(t) = \dfrac{M}{1 + Ae^{-kt}}$. It is used to model population, where M is the carrying capacity, or maximum. Let $y(t)$ be the mass of a colony of bacteria, and suppose that it is growing in a petri dish and so its maximum capacity is $M = 100$ mg, and $y(t)$ satisfies the logistic equation. Suppose that initially, there are 2 mg of bacteria and the rate of increase is 1 mg per day. [Note: The constant k is *not* equal to 1. Rather, $y'(0) = 1$].
   (a) When will the mass of bacteria be 50 mg?
   (b) What is the mass of bacteria 10 days after the mass was 2 mg?
   (c) Without using a computer (you don't need one!) Sketch this solution curve in the ty-plane, as well as the steady-state solutions $y(t) = 0$ and $y(t) = 100$.

(7) The population of a certain planet is believed to be growing according to the logistic equation (see the previous problem). The maximum population the planet can hold is 10^{10}. In year zero the population is 50% of this maximum, and the rate of increase of the population is 10^9 per year.
   (a) What is the logistic equation satisfied by the population, $y(t)$?

(b) How many years until the population reaches 90% of the maximum?
(c) Sketch this solution curve in the ty-plane, as well as the steady-state solutions $y(t) = 0$ and $y(t) = 10^{10}$.