MTHSC 208 (Differential Equations)
Dr. Matthew Macauley
HW 5
Due Friday September 11th, 2009

(1) A tank initially contains 100 gal of a salt-water solution containing 0.05 lb of salt for each gallon of water. At time zero, pure water is poured into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank that allows the salt-water solution to leave at a rate of 3 gal per minute. What will be the salt content in the tank when precisely 50 gal of salt solution remain.

(2) A tank initially contains 100 gal of pure water. Water begins entering a tank via two pipes: through pipe A at 6 gal per minute, and pipe B at 4 gal per minute. Simultaneously, a drain is opened at the bottom of the tank through which solution leaves the tank at a rate of 8 gal per minute.
   (a) To their dismay, supervisors discover that the water coming into the tank through pipe A is contaminated, containing 0.5 lb of pollutant per gallon of water. If the process had been running undected for 10 minutes, how much pollutant is in the tank at the end of this 10-minute period?
   (b) The supervisors correct their error and shut down pipe A, allowing pipe B and the drain to function in precisely the same matter as they did before the contaminant was discovered in pipe A. How long will it take the pollutant in the tank to reach one half of the level achieved in part (a)?

(3) A lake, with volume $V = 100$ km^3, is fed by a river at a rate of r km^3/yr. In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of p km^3/yr. There is another river that is fed by the lake at a rate that keeps the volume of the lake constant. This means that the rate of flow from the lake into the outlet river is $(p+r)$ km^3/yr. Let $x(t)$ denote the volume of the pollutant in the lake at time t. Then $c(t) = x(t)/V$ is the concentration of the pollutant.
   (a) Show that, under the assumption of immediate and perfect mixing of the pollutant into the lake water, the concenetration satisfies the differential equation

$$c' + \frac{p+r}{V}\,c = \frac{p}{V}\,.$$

   (b) It has been determined that a concentration of over 2% is hazardous for the fish in the lake. Suppose that $r = 50$ km^3/yr, $p = 2$ km^3/yr, and the initial concentration of pollutant in the lake is zero. How long will it take the lake to become hazardous to the health of the fish?
   (c) Suppose that the factory from parts (a) and (b) stops operating at time $t = 0$, and that the concentration of pollutant in the lake was 3.5% at that time. Approximately how long will it take before the concentration falls below 2% and the lake is no longer hazardous for the fish?

(4) Rivers do not flow at the same rate the year around. They tend to be full in the spring when the snow melts and to flow more slowly in the fall. To take this into account, suppose the flow of the input river in the previous exercise is

$$r = 50 + 20\cos(2\pi(t - 1/3))\,.$$

Our river flows at its maximum rate one-thrid into the year (i.e., around the first of April) and its its minimum around the first of October.
   (a) Setting $p = 2$, and using this flow rate, use your numerical solver, calculator, or computer, to plot the concentration for several choices of initial concentration between 0% and 4%. How would you describe the behavior of the concentration for large values of time?

(b) It might be expected that after settling into a steady state, the concentration would be greatest when the flow was smallest (i.e., around the first of October). At what time of year does it actually occur?

(5) Consider two tanks, labeled tank A and tank B for reference. Tank A contains 100 gal of solution in which is dissolved 20 lb of salt. Tank B contains 200 gal of solution in which is dissolved 40 lb of salt. Pure water flows into tank A at a rate of 5 gal/sec. There is a drain at the bottom of tank A, and the solution leaves tank A via this drain at a rate of 5 gal/sec and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 gal/sec. What is the salt content in tank B at the precise moment that tank B contains 250 gal of solution.

(6) Lake Keowee contains 1.10 km^3 of pure water. It is fed through the Jocasee dam at a rate of 0.05 km^3/yr. At time zero, there is a power plant on Lake Keowee that begins introducing a pollutant to the lake at a rate of 0.001 km^3/yr. Lake Keowee flows into Lake Hartwell at a rate of 0.051 km^3/yr. Finally, there is an outlet river from Lake Hartwell that empties into Lake Russell that keeps the volume of Lake Hartwell at a constant 3.50 km^3.

(a) Find the amount of pollutant in Lake Hartwell at the end of 3 months (i.e., $t = 0.25$ years).

(b) At the end of 3 months, the power plant closes due to mandatory furlough, and no further pollutant enters Lake Keowee. How long will it take for the pollutant in Lake Hartwell to be cut in half?