MTHSC 208 (Differential Equations)
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HW 11
Due Wednesday October 15th, 2009

For full credit, be sure to *show your work* on all of these problems!

(1) State whether the give system is autonomous or nonautonomous, and also whether it is homogeneous or nonhomogeneous.
  (a) $x' = y, \quad y' = x + 4$
  (b) $x' = x + 2y + \sin t, \quad y' = -x + y - \cos t$
  (c) $x' = -2tx + y \quad y' = 3x - y$
  (d) $x' = x + 2y + 4, \quad y' = -2x + y - 3$
  (e) $x' = 3x - y, \quad y' = x + 2y$
  (f) $x' = -x + ty, \quad y' = tx - y$
  (g) $x' = x + y + 4, \quad y' = -2x + (\sin t)y$
  (h) $x' = 3x - 4y, \quad y' = x + 3y$

(2) Transform the given 2^{nd} initial value problem into an initial value problem of two 1^{st} order equations (by letting $x_1 = u$ and $x_2 = u'$), and write it in matrix form: $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$, $\mathbf{x}(t_0) = \mathbf{x}_0$.
  (a) $u'' + 0.25u' + 4u = 2\cos 3t, \qquad u(0) = 1, \quad u'(0) = -2$
  (b) $tu'' + u' + tu = 0, \qquad u(1) = 1, \quad u'(1) = 0$

(3) In each problem below, an inhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ of two first order ODEs is given. The qualitative behavior of the solutions (as seen through their phase planes) near the critical point are all different. The motivation of this problem is to discover how the eigenvalues and eigenvectors determine the behavior of the solutions. For (a)–(e), carry out the following steps:
  (i) Find the equilibrium solution, or critical point, for the given system.
  (ii) Write the associated homogeneous equation, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and find the eigenvalues and eigenvectors of $\mathbf{A}$.
  (iii) Draw a phase portrait centered at the critical point. (The PPLANE applet at `http://math.rice.edu/~dfield/dfpp.htm` is fantastic.)
  (iv) Describe how solutions of the system behave in the vicinity of the critical point (e.g., do they approach the critial point, depart from it, spiral around it, or something else).
  (a) $x' = -x - 4y - 4, \quad y' = x - y - 6$
  (b) $x' = -0.25x - 0.75y + 8, \quad y' = 0.5x + y - 11.5$
  (c) $x' = -2x + y - 11, \quad y' = -5x + 4y - 35$
  (d) $x' = x + y - 3, \quad y' = -x + y + 1$
  (e) $x' = -5x + 4y - 35, \quad y' = -2x + y - 11$

(4) Find the general solution for each of the given system of equations. Draw a phase portrait. Describe the behavior of the solutions as $t \to \infty$.
  (a) $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$ (b) $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$
  (c) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$ (d) $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}$

(5) In each of the next four problems, the eigenvalues and eigenvectors of a matrix A are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Without using a computer, draw each of the following graphs.
  (i) Sketch a phase portrait of the system.
  (ii) Sketch the trajectory passing through the initial point $(2, 3)$.
  (iii) For the trajectory in part (ii), sketch the component plots of x_1 versus t and x_2 versus t on the same set of axes.

(a) $\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = -4, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

(b) $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = -4, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

(c) $\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = 4, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

(d) $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = 4, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

(6) In each of the next four problems, the eigenvalues and eigenvectors of a matrix A are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Without using a computer, draw each of the following graphs.

(i) Sketch a phase portrait of the system.

(ii) Sketch the trajectory passing through the initial point $(2, 3)$.

(a) $\lambda_1 = -4, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = -1, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

(b) $\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = -1, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

(c) $\lambda_1 = -4, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = 1, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

(d) $\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = 1, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$