MTHSC 208 (Differential Equations)
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HW 12
Due Wednesday October 21st, 2009

For full credit, be sure to *show your work* on all of these problems!

(1) Find the general solution for each of the given systems in terms of real-valued function, and draw a phase portrait. Describe the behavior of the solutions as $t \to \infty$.
(a) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$ (b) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$ (c) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

(2) In the problems below, the coefficient matrix contains a parameter α.
(a) Determine the eigenvalues in terms of α.
(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.
(c) Draw a phase portrait for a value of α slight below, and for another value slightly above, each critical value.
(d) Draw a phase portrait when α is exactly the critical value.
(a) $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$ (b) $\mathbf{x}' = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix} \mathbf{x}$

(3) Find the general solution for each of the given systems and draw a phase portrait. Describe the behavior of the solutions as $t \to \infty$.
(a) $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ (b) $\mathbf{x}' = \begin{pmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{pmatrix} \mathbf{x}$ (c) $\mathbf{x}' = \begin{pmatrix} -1 & -1/2 \\ 2 & -3 \end{pmatrix} \mathbf{x}$

(4) Consider functions $x(t)$ and $y(t)$, and the linear system

$$x' = a_{11}x + a_{12}y, \quad y' = a_{21}x + a_{22}y,$$

where $a_{11}, \ldots, a_{22}$ are real constants. Let $p = a_{11} + a_{22}$, $q = a_{11}a_{22} - a_{12}a_{21}$, and $\Delta = p^2 - 4q$. Observe that p and q are the trace and determinant, respectively, of the coefficient matrix of the given system. Show that the critical point of $(0, 0)$ is a
(a) Node if $q > 0$ and $\Delta \geq 0$;
(b) Saddle point if $q < 0$;
(c) Spiral point if $p \neq 0$ and $\Delta < 0$;
(d) Center if $p = 0$ and $q > 0$.
Hint: The conclusions in (a)–(d) can be obtained by studying the eigenvalues λ_2 and λ_2. It may also be helpful to establish the relations that $\det A = \lambda_1\lambda_2 = p$ and $\operatorname{tr} A = \lambda_1 + \lambda_2 = q$.

(5) Continuing Problem 4, show that the critical point $(0, 0)$ is
(a) Asymptotically stable if $q > 0$ and $p < 0$;
(b) Stable if $q > 0$ and $p = 0$;
(c) Unstable if $q < 0$ and $p > 0$.

(6) Draw the parabola $p^2 - 4q = 0$ in the pq-plane, and sketch the regions in the plane corresponding to the cases in #4(a)–(d), and #5(a)–(c).