MTHSC 208 (Differential Equations)
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HW 16
Due Tuesday November 10th, 2009

For full credit, be sure to *show your work* on all of these problems!

(1) For each initial value problem, sketch the forcing term, and then solve for $y(t)$. Recall that the function $H_{ab}(t) = H(t-a) - H(t-b)$ is the interval function.
  (a) $y'' + 4y = H_{01}(t), \quad y(0) = 0, \ y'(0) = 0$
  (b) $y'' + 4y = t\,H_{01}(t), \quad y(0) = 0, \ y'(0) = 0$

(2) Define the function
$$\delta_p^\epsilon(t) = \frac{1}{\epsilon}\left(H_p(t) - H_{p+\epsilon}(t)\right).$$
  (a) Show that the Laplace transform of $\delta_p^\epsilon(t)$ is given by
$$\mathcal{L}\left\{\delta_p^\epsilon(t)\right\} = e^{-sp}\,\frac{1-e^{-s\epsilon}}{s\epsilon}.$$
  (b) Use l'Hôpital's rule to take the limit of the result in part (a) as $\epsilon \to 0$. How does this result agree with the fact that $\mathcal{L}\{\delta_p(t)\} = e^{-sp}$?

(3) Solve the following initial value problems.
  (a) $y'' + 4y = \delta(t), \quad y(0) = 0, \ y'(0) = 0$
  (b) $y'' - 4y' - 5y = \delta(t), \quad y(0) = 0, \ y'(0) = 0$

(4) (a) Consider the initial value problem
$$x'' + 2x' + 2x = \delta(t), \quad x(0) = x'(0) = 0.$$
  (a) Use the fact that $\mathcal{L}\{\delta(t)\}(s) = 1$ to show that the solution is $x(t) = e^{-t}\sin t$ for $t \geq 0$.
  (b) Show that the solution of
$$x'' + 2x' + 2x = \delta_0^\epsilon(t), \quad x(0) = x'(0) = 0$$
  is
$$x_\epsilon(t) = \frac{1}{2\epsilon}\begin{cases} 1 - e^{-t}(\cos t + \sin t), & 0 \leq t < \epsilon; \\ -e^{-t}(\cos t + \sin t) + e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon)), & t \geq \epsilon. \end{cases}$$
  (c) Use l'Hôpital's rule to argue that the solution of part (b) approaches that of part (a) as $\epsilon \to 0$, at least for $t > 0$.

(5) Define the function
$$H_p^\epsilon(t) = \begin{cases} 0, & 0 \leq t < p \\ \frac{1}{\epsilon}(x-p), & p \leq t < p+\epsilon \\ 1, & t \geq p+\epsilon \end{cases}$$
  (a) Sketch the graph of $H_p^\epsilon(t)$.
  (b) Without being too precise about things, we could argue that $H_p^\epsilon(t) \to H_p(t)$ as $\epsilon \to 0$, where $H_p(t) = H(t-p)$. Sketch the graph of the derivative of $H_p^\epsilon(t)$.
  (c) Compare your result in (b) with the graph of $\delta_p^\epsilon(t)$. Argue that $H_p'(t) = \delta_p(t)$.

(6) Use the fact that $\mathcal{L}\{\delta_p(t)\}(s) = e^{-ps}$ to show that the solution of the equation
$$x' = \delta_p(t), \qquad x(0) = 0$$
is $x(t) = H_p(t)$, giving further credence to the argument in the previous exercise that the "derivative of a unit step is a unit impulse," as engineers like to say.