MTHSC 208 (Differential Equations)
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HW 20
Due Tuesday November 23rd, 2009

(1) Consider the ODE $y'' = 4y$. We know that the general solution is $y(t) = C_1e^{2t} + C_2e^{-2t}$, i.e., $\{e^{2t}, e^{-2t}\}$ is a basis for the solution space. Use the fact that $e^x = \cosh x + \sinh x$ to find two distinct solutions involving hyperbolic sines and cosines. Write the general solution using these functions.

(2) We will find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = 5\sin x + 3\sin 2x.$$

(a) Assume that $u(x,t) = f(x)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ). Solve for $g(t)$, $f(x)$, and λ.
(b) Using your solution to (a) and the principle of superposition, find the general solution to the PDE
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
subject to the *Dirichlet boundary conditions*:
$$u(0,t) = u(\pi,t) = 0 .$$
(c) Solve the initial value problem, i.e., find the particular solution $u(x,t)$ that satisfies $u(x,0) = 5\sin x + 3\sin 2x$.
(d) What is the steady-state solution, i.e., $\lim_{t\to\infty} u(x,t)$?

(3) Consider a similar situation as the previous problem, but with slightly different boundary and initial conditions.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 30, \quad u(\pi,t) = 100$$
$$u(x,0) = 30 + \tfrac{70}{\pi}x + 5\sin x + 3\sin 2x.$$

(a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.
(b) Use your physical intuition to determine what the steady-state solution $u_{ss}(x,t)$ is.
(c) Write down the solution to this initial value problem by adding the steady-state solution to the solution you derived in the preveious problem.
(d) How does this compare to the structure of the solution to the ODE for Newton's law of heating / cooling? [*Hint*: Consider an example, e.g., $T(t) = 72 + T_h(t) = 72 + Ce^{-kt}$. Note that the heat equation is the 1-dimensional analog of Newtons law of heating / cooling (which is typically applied to a point-mass, or a "0-dimensional" object).]

(4) We will find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u_x(0,t) = u_x(\pi,t) = 0, \qquad u(x,0) = x(\pi - x).$$

(a) Describe (and sketch) a physical situation that this models. Be sure to describe the significance of the boundary and initial conditions.
(b) Assume that $u(x,t) = f(x)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ). Solve for $g(t)$, $f(x)$, and λ.
(c) Using your solution to (b), find the general solution to the PDE
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the *Neumann boundary conditions*:

$$u_x(0,t) = u_x(\pi,t) = 0 \ .$$

(d) Finally, solve the initial value problem, i.e., find the particular solution $u(x,t)$ that satisfies $u(x,0) = x(\pi - x)$.

(e) What is the steady-state solution?

(5) Let $u(x,t)$ be the temperature of a bar of length 10, that is insulated so that no heat can enter or leave. Suppose that initially, the temperature increases linearly from 70° at one endpoint, to 80° at the other endpoint.

(a) Sketch the initial heat distribution on the bar, and express it as a function of x.

(b) Write down an initial value problem (a PDE with boundary and initial conditions) to which $u(x,t)$ is a solution (Let the constant from the heat equation be c^2).

(c) What will the steady-state solution be?

(6) Consider the following PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad u(x,0) = 3\sin\frac{5x}{2}.$$

(a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.

(b) Assume that the solution is of the form $u(x,t) = f(x)g(t)$, and plug this into the PDE to get the eigenvalue problem (set equal to a constant λ). From this, write down two ODEs; one for f and one for g. Include boundary conditions for f.

(c) Solve the ODEs from the previous part for f and g. You may assume that $\lambda = -\omega^2$, (i.e., that $\lambda < 0$). Determine ω (be sure to show your work for this part, the answer may surprise you!).

(d) Write down the general solution for $u(x,t)$, which solve the *mixed boundary conditions*:

$$u(0,t) = u_x(\pi,t) = 0 \ .$$

(e) Find the particular solution for $u(x,t)$ satisfying the initial condition $u(x,0) = 3\sin(5x/2)$.

(f) What is the steady-state solution?