Consider the ODE \( y'' = 4y \). We know that the general solution is \( y(t) = C_1 e^{2t} + C_2 e^{-2t} \), i.e., \( \{e^{2t}, e^{-2t}\} \) is a basis for the solution space. Use the fact that \( e^x = \cosh x + \sinh x \) to find two distinct solutions involving hyperbolic sines and cosines. Write the general solution using these functions.

We will find the function \( u(x, t) \), defined for \( 0 \leq x \leq \pi \) and \( t \geq 0 \), which satisfies the following conditions:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = 5 \sin x + 3 \sin 2x.
\]

(a) Assume that \( u(x, t) = f(x)g(t) \). Plug this back into the PDE and separate variables to get the eigenvalue problem (set equal to a constant \( \lambda \)). Solve for \( g(t) \), \( f(x) \), and \( \lambda \).

(b) Using your solution to (a) and the principle of superposition, find the general solution to the PDE

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]

subject to the Dirichlet boundary conditions:

\[
u(0, t) = u(\pi, t) = 0.
\]

(c) Solve the initial value problem, i.e., find the particular solution \( u(x, t) \) that satisfies \( u(x, 0) = 5 \sin x + 3 \sin 2x \).

(d) What is the steady-state solution, i.e., \( \lim_{t \to \infty} u(x, t) \)?

Consider a similar situation as the previous problem, but with slightly different boundary and initial conditions.

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = 30, \quad u(\pi, t) = 100
\]

\[
u(x, 0) = 30 + \frac{70}{\pi} x + 5 \sin x + 3 \sin 2x.
\]

(a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of both boundary conditions and the initial condition.

(b) Use your physical intuition to determine what the steady-state solution \( u_{ss}(x, t) \) is.

(c) Write down the solution to this initial value problem by adding the steady-state solution to the solution you derived in the previous problem.

(d) How does this compare to the structure of the solution to the ODE for Newton’s law of heating / cooling? [Hint: Consider an example, e.g., \( T(t) = 72 + T_h(t) = 72 + Ce^{-kt} \). Note that the heat equation is the 1-dimensional analog of Newtons law of heating / cooling (which is typically applied to a point-mass, or a “0-dimensional” object).]

We will find the function \( u(x, t) \), defined for \( 0 \leq x \leq \pi \) and \( t \geq 0 \), which satisfies the following conditions:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u_x(0, t) = u_x(\pi, t) = 0, \quad u(x, 0) = x(\pi - x).
\]

(a) Describe (and sketch) a physical situation that this models. Be sure to describe the significance of the boundary and initial conditions.

(b) Assume that \( u(x, t) = f(x)g(t) \). Plug this back into the PDE and separate variables to get the eigenvalue problem (set equal to a constant \( \lambda \)). Solve for \( g(t) \), \( f(x) \), and \( \lambda \).

(c) Using your solution to (b), find the general solution to the PDE

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]
subject to the Neumann boundary conditions:
\[ u_x(0,t) = u_x(\pi, t) = 0 . \]
(d) Finally, solve the initial value problem, i.e., find the particular solution \( u(x,t) \) that satisfies \( u(x,0) = x(\pi - x) \).
(e) What is the steady-state solution?

(5) Let \( u(x,t) \) be the temperature of a bar of length 10, that is insulated so that no heat can enter or leave. Suppose that initially, the temperature increases linearly from 70° at one endpoint, to 80° at the other endpoint.
(a) Sketch the initial heat distribution on the bar, and express it as a function of \( x \).
(b) Write down an initial value problem (a PDE with boundary and initial conditions) to which \( u(x,t) \) is a solution (Let the constant from the heat equation be \( c^2 \)).
(c) What will the steady-state solution be?

(6) Consider the following PDE:
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) = 0, \quad u(x, 0) = 3 \sin \frac{5x}{2} . \]
(a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of both boundary conditions and the initial condition.
(b) Assume that the solution is of the form \( u(x,t) = f(x)g(t) \), and plug this into the PDE to get the eigenvalue problem (set equal to a constant \( \lambda \)). From this, write down two ODEs; one for \( f \) and one for \( g \). Include boundary conditions for \( f \).
(c) Solve the ODEs from the previous part for \( f \) and \( g \). You may assume that \( \lambda = -\omega^2 \), (i.e., that \( \lambda < 0 \)). Determine \( \omega \) (be sure to show your work for this part, the answer may surprise you!).
(d) Write down the general solution for \( u(x,t) \), which solve the mixed boundary conditions:
\[ u(0, t) = u_x(\pi, t) = 0 . \]
(e) Find the particular solution for \( u(x,t) \) satisfying the initial condition \( u(x,0) = 3 \sin(5x/2) \).
(f) What is the steady-state solution?