MTHSC 208 (Differential Equations)
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HW 21
Due Tuesday December 1st, 2009

(1) Let $u(x,t)$ be the temperature of a bar of length 10, at position x and time t (in hours). Suppose that initially, the temperature increases linearly from 70° at the left endpoint to 80° at the other end. Furthermore, suppose that the temperature at the left end of the bar is held at a constant 70 degrees, and that the right end is insulated so no heat can escape. Finally, suppose that the interior of the bar is poorly insulated, so heat escapes from it, causing the temperature to decrease at and additional constant rate of 1° per hour. Write an initial value problem for $u(x,t)$ that could model this situation.

(2) Consider the following PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 0, \quad u_x(\pi,t) + \gamma\, u(\pi,t) = 0, \qquad u(x,0) = h(x)\ ,$$

where γ is a non-negative constant.

(a) Describe a physical situation that this models. Be sure to describe the impact of the initial condition, *both* boundary conditions and the constant γ.
(b) What is the steady-state solution, and why?

(3) We will solve for $u(\theta,t)$, the two-variable function that describes the temperature of a circular wire, and satisfies the heat equation with *periodic boundary conditions*:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \qquad u(\theta+2\pi,t) = u(\theta,t), \qquad u(\theta,0) = 2 + 4\sin 3\theta - \cos 5\theta.$$

(a) Assume that $u(\theta,t) = f(\theta)g(t)$. Plug this back into the PDE, separate variables, and set the equation equal to a constant λ. Write down two ODEs: one for $f(\theta)$ and one for $g(t)$. *Note: The notation is easier when you write the derivatives in the numerator, e.g.,* $g'(t)/g(t) = \lambda$*, rather than* $g(t)/g'(t) = \lambda$.
(b) Use the periodic boundary conditions for $u(\theta,t)$ to derive similar periodic boundary conditions for $f(\theta)$. Solve for λ and $f(\theta)$. *Note: You won't be able to conclude that* $a = 0$ *or* $b = 0$ *– so unlike before, they'll both stick around.*
(c) Now that you know λ, solve the ODE for $g(t)$.
(d) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(\theta,t) = f_n(\theta)g_n(t)$.
(e) Find the particular solution to the initial value problem (by plugging in $t = 0$).
(f) What is the steady-state solution? Give a mathematical *and* intuitive (physical) justification for this.

(4) Find the function $u(x,t)$ defined for $0 \le x \le \pi$ and $t \ge 0$ which satisfies the following initial value problem of the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 8\sin x + 11\sin 2x + 15\sin 4x, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

(a) Assume that $u(x,t) = f(x)g(t)$. Plug this back into the PDE, separate variables, and set the equation equal to a constant λ. Write down two ODEs: one for $f(x)$ and one for $g(t)$.
(b) Use the periodic boundary conditions for $u(x,t)$ to derive similar periodic boundary conditions for $f(x)$. Solve for λ and $f(x)$.
(c) Now that you know λ, solve the ODE for $g(t)$.

(d) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(x,t) = f_n(x)g_n(t)$.

(e) Find the particular solution to the initial value problem by using the initial conditions.

(5) In this problem, we will find the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = x(\pi - x), \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Steps (a)–(d) are the same as in the previous problem, and need not be repeated. Instead, repeat part (e) with these new initial conditions. What physical situation does this model? Give a physical interpretation for both boundary conditions, and both initial conditions, and sketch this scenerio at time $t = 0$.

(6) In this problem, we will find the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies different initial conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = x(\pi - x).$$

Steps (a)–(d) are the same as in Problem (4), and need not be repeated. Instead, repeat part (e) with these new initial conditions. What physical situation does this model? Give a physical interpretation for both boundary conditions, and both initial conditions, and sketch this scenerio at time $t = 0$.