

**MTHSC 208 (Differential Equations)**

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**HW 22**

**Due Friday December 4th, 2009**

- (1) Which of the following functions are harmonic?

- (a)  $f(x) = 10 - 3x$ .
- (b)  $f(x, y) = x^2 + y^2$ .
- (c)  $f(x, y) = x^2 - y^2$ .
- (d)  $f(x, y) = e^x \cos y$ .
- (e)  $f(x, y) = x^3 - 3xy^2$ .

- (2) (a) Solve the following Dirichlet problem for Laplace's equation in a square region: Find  $u(x, y)$ ,  $0 \leq x \leq \pi$ ,  $0 \leq y \leq \pi$  such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(0, y) = u(\pi, y) = 0,$$

$$u(x, 0) = 0, \quad u(x, \pi) = \frac{1}{5} \sin x - \frac{2}{5} \sin 2x + \frac{3}{5} \sin 3x.$$

- (b) Solve the following Dirichlet problem for Laplace's equation in the same square region: Find  $u(x, y)$ ,  $0 \leq x \leq \pi$ ,  $0 \leq y \leq \pi$  such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(0, y) = 0, \quad u(\pi, y) = y(\pi - y),$$

$$u(x, 0) = u(x, \pi) = 0$$

- (c) By adding the solutions to parts (a) and (b) together, find the solution to the Dirichlet problem: Find  $u(x, y)$ ,  $0 \leq x \leq \pi$ ,  $0 \leq y \leq \pi$  such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(0, y) = 0, \quad u(\pi, y) = y(\pi - y),$$

$$u(x, 0) = 0, \quad u(x, \pi) = \frac{1}{5} \sin x - \frac{2}{5} \sin 2x + \frac{3}{5} \sin 3x.$$

- (d) Sketch the solutions to (a), (b), and (c). *Hint: it is enough to sketch the boundaries, and then use the fact that the solutions are harmonic functions.*

- (e) Consider the heat equation in a square region, along with the following boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad u(0, y) = 0, \quad u(\pi, y) = y(\pi - y),$$

$$u(x, 0) = 0, \quad u(x, \pi) = \frac{1}{5} \sin x - \frac{2}{5} \sin 2x + \frac{3}{5} \sin 3x.$$

What is the steady-state solution? (Note: This will *not* depend on the initial conditions!)

- (3) Solve the following initial value problem for the heat equation in a square region: Find  $u(x, y, t)$ , where  $0 \leq x \leq \pi$ ,  $0 \leq y \leq \pi$  and  $t \geq 0$  such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0$$

$$u(x, y, 0) = 2 \sin x \sin y + 5 \sin 2x \sin y.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$f_{xx} + f_{yy} = \lambda f, \quad f(x, 0) = f(x, \pi) = f(0, y) = f(\pi, y) = 0,$$

are of the form

$$\lambda = -(m^2 + n^2), \quad f(x, y) = b_{mn} \sin mx, \sin ny,$$

for  $m = 1, 2, 3, \dots$  and  $n = 1, 2, 3, \dots$ , where  $b_{mn}$  is a constant.

- (4) Solve the following initial value problem for the heat equation in a square region: Find  $u(x, y, t)$ , where  $0 \leq x \leq \pi$ ,  $0 \leq y \leq \pi$  and  $t \geq 0$  such that

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ u(x, 0, t) &= u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\ u(x, y, 0) &= (7 \sin x) y(\pi - y). \end{aligned}$$

Sketch the initial heat distribution over this region. What is the steady-state solution?

- (5) (a) Find the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to the boundary conditions

$$\begin{aligned} u(x, 0, t) &= u(0, y, t) = u(\pi, y, t) = 0 \\ u(x, \pi, t) &= x(\pi - x). \end{aligned}$$

- (b) Sketch the steady-state solution.

- (6) Solve the following initial value problem for a vibrating square membrane: Find  $u(x, y, t)$ ,  $0 \leq x \leq \pi$ ,  $0 \leq y \leq \pi$  such that

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\ u(x, 0, t) &= u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\ u(x, y, 0) &= p(x)q(y), \quad \frac{\partial u}{\partial t}(x, y, 0) = 0. \end{aligned}$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \leq x \leq \pi/2, \\ \pi - x, & \text{for } \pi/2 \leq x \leq \pi, \end{cases}, \quad p(y) = \begin{cases} y, & \text{for } 0 \leq y \leq \pi/2, \\ \pi - y, & \text{for } \pi/2 \leq y \leq \pi. \end{cases}$$

Sketch the initial displacement of the square membrane. What is the long-term behavior of  $u(x, y, t)$ ?