MTHSC 208, WEEK 1

<u>Q</u>: What is a Differential Equation?

<u>A</u>: An equation involving a function & its derivatives.

<u>Examples</u>:

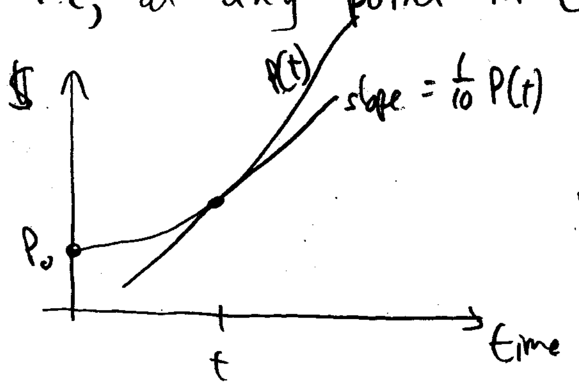
- $\boxed{\text{Finance}}$ The $\overbrace{\text{rate of growth of an investment}}^{P'(t)}$ is proportional to the $\underbrace{\text{amount of the investment}}_{P(t)}$.

$P'(t) = r\,P(t)$ (often, just write $P' = rP$).

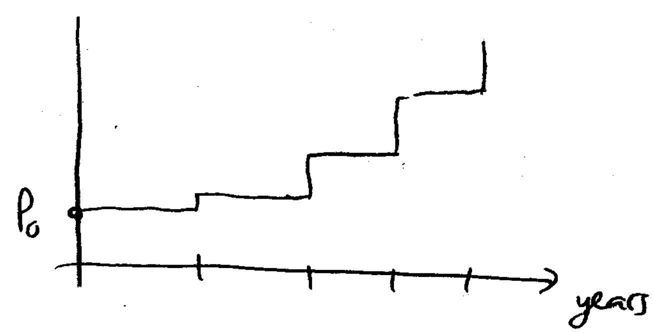
e.g., A mutual fund is increasing at a 10% rate.

$P'(t) = \frac{1}{10} P(t)$ (or $P' = \frac{1}{10}P$).

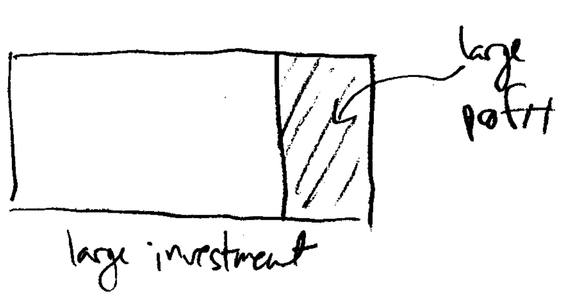
<u>Note</u>: We assume that interest is compounded continuously, i.e, at any point in time, the rate of change is $\frac{1}{10}P$.

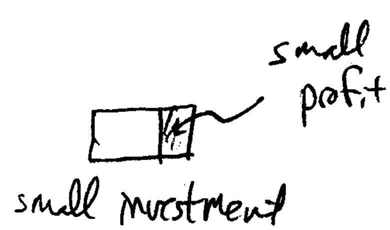


vs.



<u>Big idea</u>: Rate of change of a function is proportional to the function itself: $f' = rf$





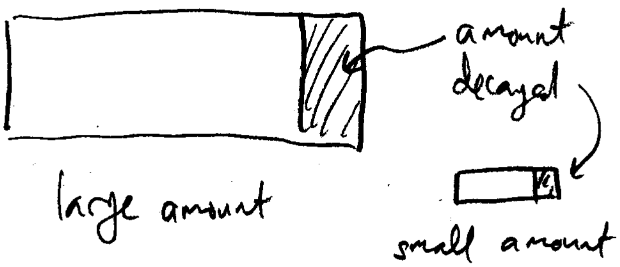
- **Biology** A colony of rabbits grows at a rate proportional to its size.

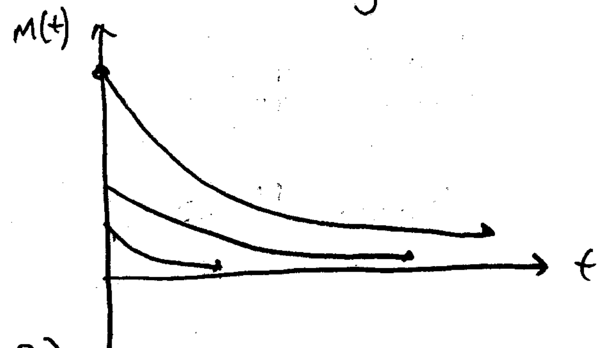
  $P'(t) = k\, P(t)$ Note: $k > 0$ (why?)

  Note: It can't keep doubling forever. This is just a model, good for small t.

  e.g., 2 rabbits, 4 rabbits, 8 rabbits, 16 rabbits, etc...

- **Chemistry** A radioactive substance decays at a rate proportional to how much is remaining.





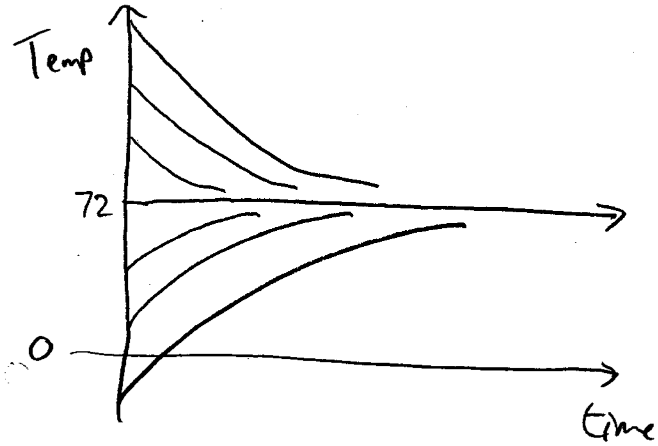
$m'(t) = k\, m(t)$ Note: $k < 0$ (why?)

Sample question: If there are 30 grams initially, and 20 grams after one year, what is the half-life?

Think: • Is it even clear that "half-life" is well-defined?
• Compare this to investments

- **Physics** The temperature of a cup of coffee cools at a rate proportional to: "(temp of coffee) - (ambient temp)."

Think: Imagine putting a cup of 75° water, and a cup of 200° water in a 72° room



$T'(t) = k(72 - T(t))$, $k > 0$.

This is decay towards a limiting value.

$T(t) = 72$ is a steady-state (const.) soln.

What else exhibits this behavior in nature (approximately)

- Earth's population
- Velocity of a falling object with air resistance.

(here, "terminal velocity" plays the role of air resistance).

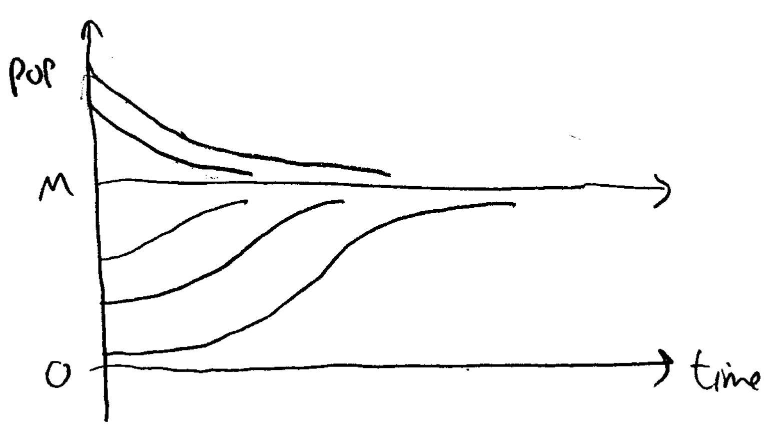
But notice! Population growth is a little different.

when population is small, it grows exponentially.

when population is large, it "decays → carrying capacity"

How do we put these two together?

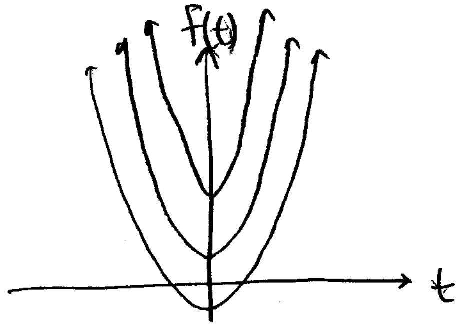
Ans: Logistic equation: $P'(t) = \underbrace{r\left(1 - \frac{P(t)}{M}\right)}_{\text{decay} \to M} \underbrace{P(t)}_{\text{exp. growth}}$



Recall integral calculus.

Q: what is the antiderivative of $f(t) = 2t$?

A: $F(t) = t^2 + C$



All of these have derivative $f(t) = 2t$.

Graphically:

Q: The velocity of a car is $x'(t) = 2t$.
How far from home is it after t hrs?

A: $x(t) = t^2 + C$ ← initial distance from home. ("initial condition")

An investment takes 5 years to double.

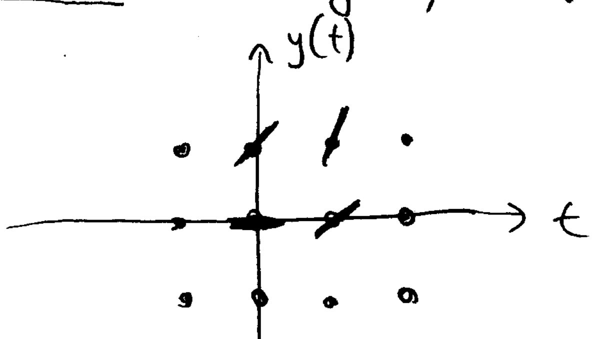
Q: How much do we have after 8 years?

A: We don't know until we specify how much we had initially.

Plotting solutions to "ordinary differential equations (ODEs)"

Consider the ODE: $y' = 2y + t$.
We don't know how to solve it (yet), but we can still "see" the sol'ns.

Method 1: On a grid, draw the "slope field" point-by-point



slope → $y' = 2y + t$

$(0,0)$: $y' = 0$
$(0,1)$: $y' = 2$
$(1,0)$: $y' = 1$
$(1,1)$: $y' = 3$

* Downside: TEDIOUS!!!

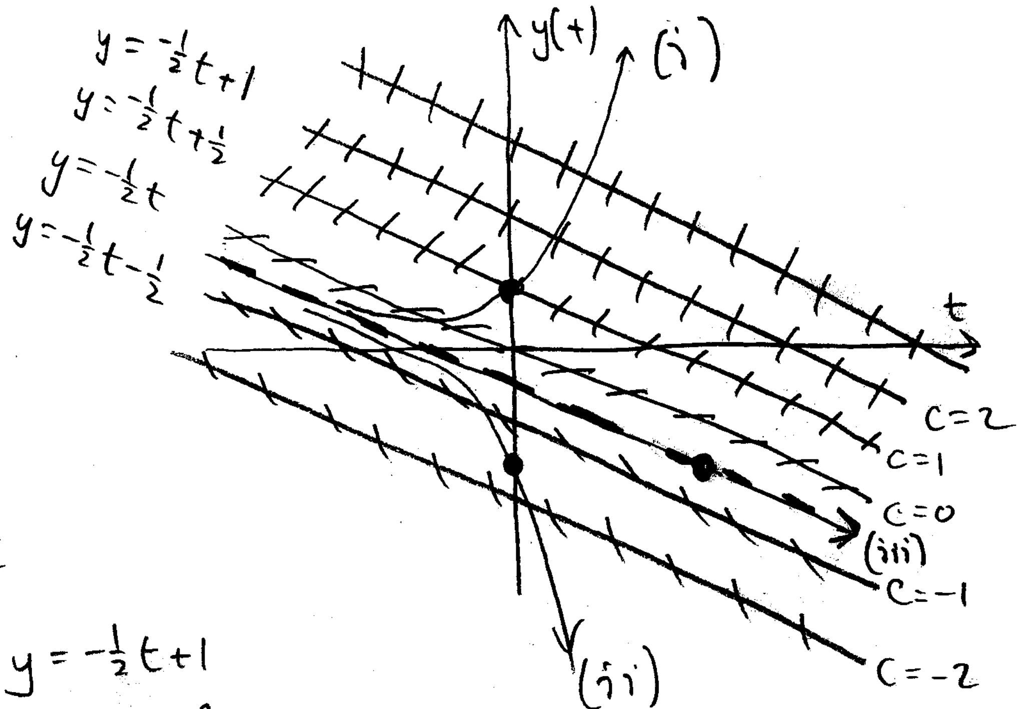
Method 2: Isoclines. (Used to sketch the slope field of an ODE).

Def: An isocline is a line or curve on which $y' = \text{const}$.

Q: when is $y' = 0$?

A: When $2y + t = 0$.

i.e., $y = -\frac{1}{2}t$



Q: when is $y' = 1$?

A: When $2y + t = 1$

i.e., $y = -\frac{1}{2}t + \frac{1}{2}$

Continue... $y' = 2 \Rightarrow y = -\frac{1}{2}t + 1$

$y' = 3 \Rightarrow y = -\frac{1}{2}t + \frac{3}{2}$

$y' = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}t - \frac{1}{4}$

Exercise: Sketch the solution curves satisfying:

(i) $y(0) = 1$, (ii) $y(0) = -\frac{3}{4}$ (iii) $y(1) = -\frac{3}{4}$

Example: Sketch the solutions of $y' = y^2$

$y' = 0 \Rightarrow y = 0$

$y' = 1 \Rightarrow y = \pm 1$

$y' = 4 \Rightarrow y = \pm 2$

$y' = -1 \Rightarrow \phi$.