# MTHSC 208, WEEK 2

Week 1 summary:

- In many real-world situations, there are simple relations between a function and its derivatives. These can be expressed as ODE's.
- Exponential growth: $y' = ky \quad k > 0$
  Exponential decay: $y' = ky \quad k < 0$
  Decay $\rightarrow$ value: $y' = k(A-y) \quad k < 0$.
- Slope fields: a way to "visualize" all solutions to an ODE. We can sketch a slope field using isoclines (not in textbook!)
  Set $y' = \text{const}$, plot the resulting line/curve.

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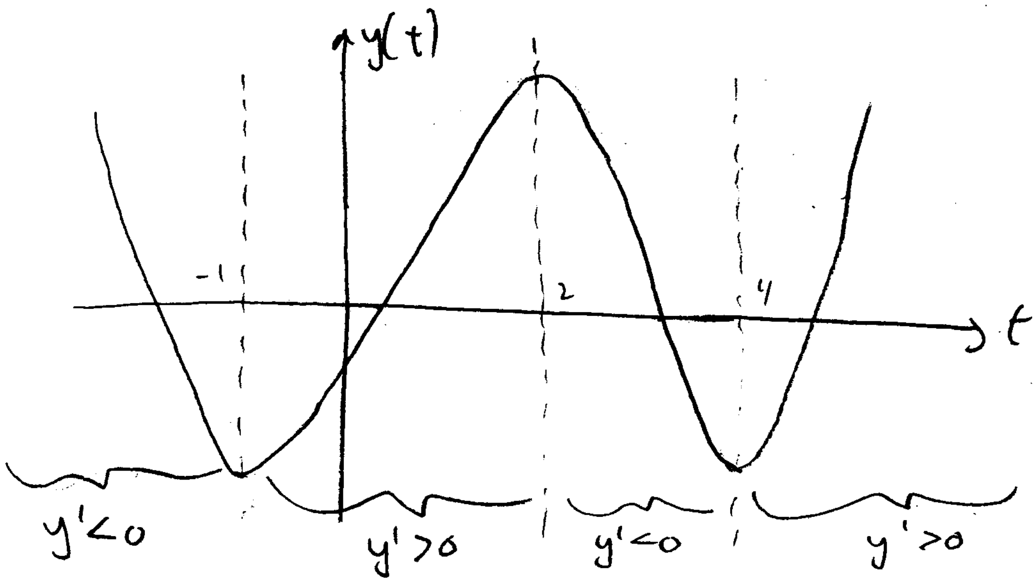
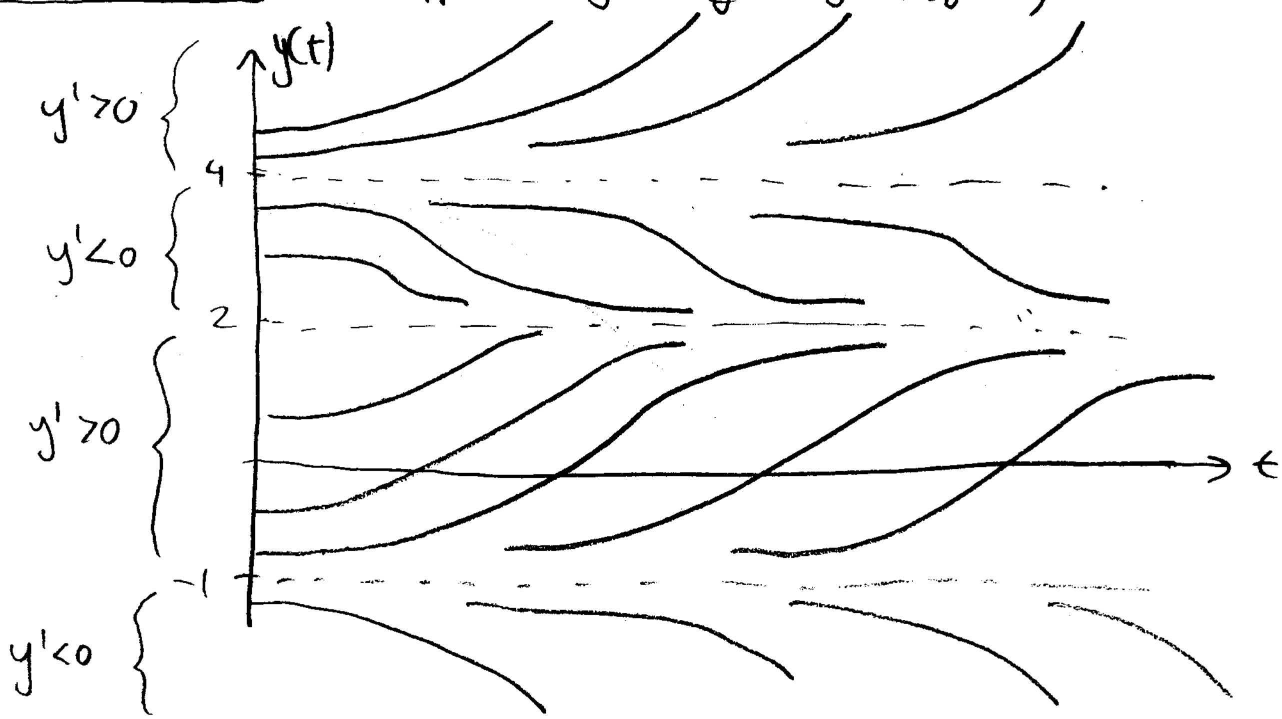
Plotting slope fields.

Method 3: A shortcut when the ODE is autonomous ($y'$ doesn't depend on $t$).

Recall basic calculus: Suppose $y'(t) = (t+1)(t-2)(t-4)$



Now, in ODEs: Suppose $y' = (y+1)(y-2)(y-4)$




* $y'$ does not depend on $t$ (it is autonomous).

This was even easier than using isoclines.

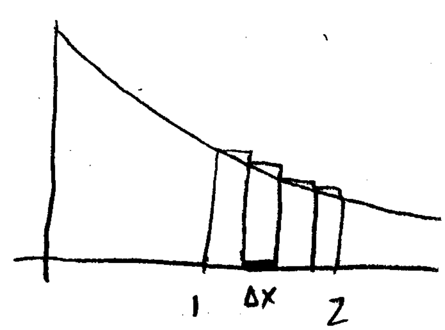
Approximating solutions to ODEs:

Question: Consider the ODE $y' = y - t$, and say $y(1) = 1$.

Can we approximate $y(1.5)$?

Analogy 1: I give you 2 hrs to compute $\int_1^2 e^{-x^2}\,dx$ to 3 decimal places. Could you do it, and how?

You'd probably use Riemann sums.
But what would you use for $\Delta x$?
Would 0.1 work? 0.01? 0.001?



Analogy 2: Compute $e^2$ to 10 decimal places.

How would you do this (say you only know $e$ to a few decimal places).

Use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

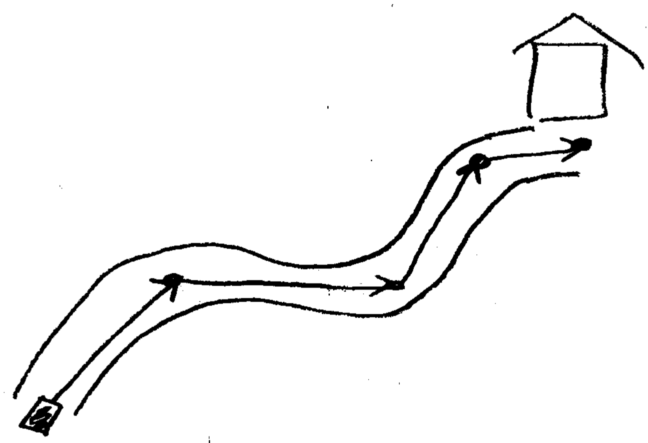
But how many terms would you need to keep?

There are questions in Numerical analysis.

Now, back to our question. We will use Euler's method.

Pictorially:

Suppose we want to steer a robot down a winding path.



Revisiting our example: $y' = y - t$, $y(1) = 1$. Let $h = 0.1$ ← our "step-size"

Start at $(t_0, y_0) = (1,1)$.

Compute $y' = y - t = 1 - 1 = 0$.

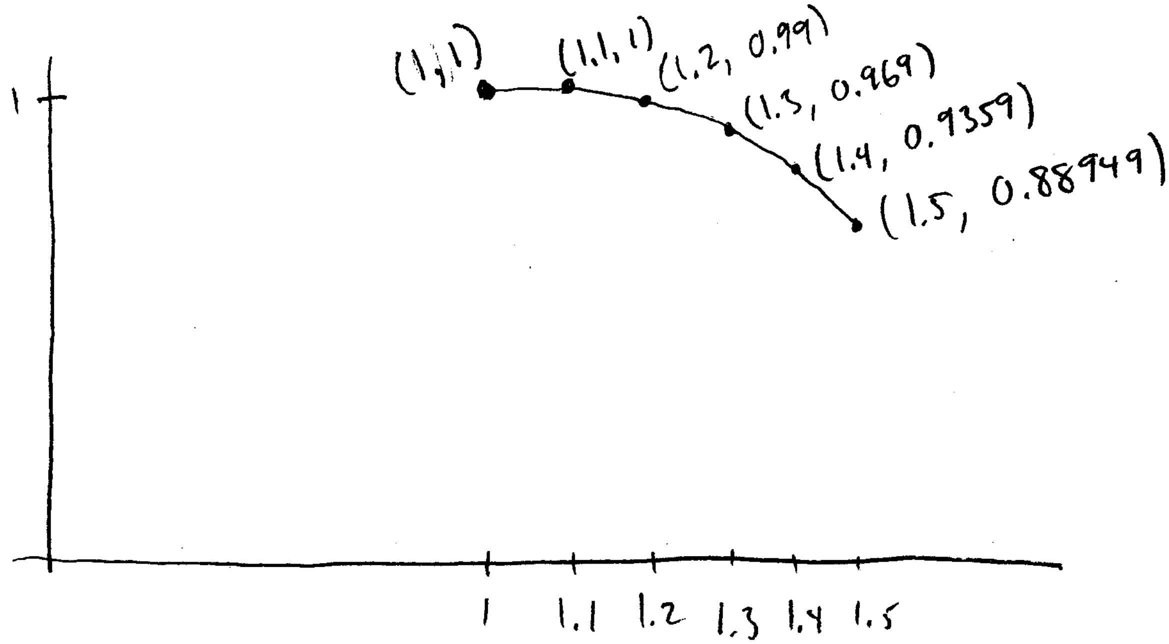
Sketch the segment with slope $m = 0$.

Now, we're at $(t_1, y_1) = (1.1,\ 1 + 0h) = (1.1, 1)$.

Compute $y' = y - t = 1 - 1.1 = -0.1$

Sketch the segment with slope $m = -0.1$.

$\boxed{4}$




Now, we're at $(t_2, y_2) = (1.2,\ 1 + mh) = (1.2,\ 1 + (-\frac{1}{10})(\frac{1}{10})) = (1.2, 0.99)$

Recompute slope: $y' = y - t = 0.99 - 1.2 = -0.21.$

Now, we're at $(t_3, y_3) = (t_2 + h,\ y_2 + mh) = (1.3,\ 0.99 + (-0.21)(\frac{1}{10}))$
$= (1.3,\ 0.969).$

Recompute slope: $y' = y - t = 0.969 - 1.3 = -0.331.$

Now, we're at $(t_4, y_4) = (t_3 + h,\ y_3 + mh) = (1.4,\ 0.969 + (-0.331)(\frac{1}{10}))$
$= (1.4,\ 0.9359).$

Recompute slope: $y' = y - t = 0.9359 - 1.4 = -0.4641.$

Now, we're at $(t_5, y_5) = (t_4 + h,\ y + mh) = (1.5,\ 0.9359 + (-0.4641)(\frac{1}{10}))$
$= (1.5,\ 0.88949).$

* We've determined that $\boxed{y(1.5) \approx 0.88949}$

Note: The actual value is $y(1.5) = -e^{0.5} + 2.5 \approx 0.85128.$

## Euler's method (summary)

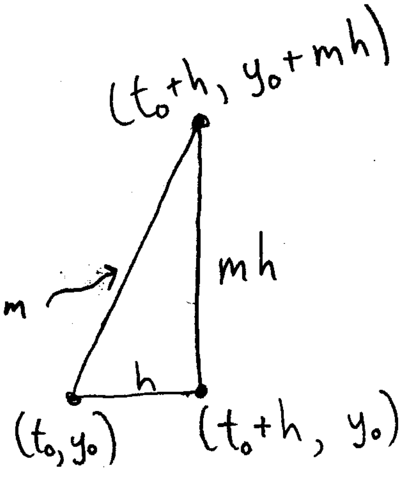
Given $y' = f(t, y)$, $y(t_0) = y_0$, step size $h$.

$(t_1, y_1) = (t_0 + h,\ y_0 + f(t_0, y_0)\cdot h)$

$(t_2, y_2) = (t_1 + h,\ y_1 + f(t_1, y_1)\cdot h)$

$\vdots$

$(t_{k+1}, y_{k+1}) = (t_k + h,\ y_k + f(t_k, y_k)\cdot h)$




## Solving ODEs

Like integration, sometimes there's a method, other times it's "art."

Example: Find all solutions to $y' = ky \Rightarrow \frac{dy}{dt} = kt$.

"Magic": Multiply thru by $dt$: $dy = ky\, dt$

Divide thru by $y$ & integrate: $\int \frac{1}{y}\, dy = \int k\, dt$

$$\ln y = kt + c$$

Take exp. of both sides $\quad y = e^{kt+c}$

$$= e^c e^{kt}$$

let $C = e^c$ $\quad \boxed{y(t) = Ce^{kt}}$

Q: what is $C$?

A: $y(0)$. "initial condition"

This technique is called <u>separation of variables.</u>

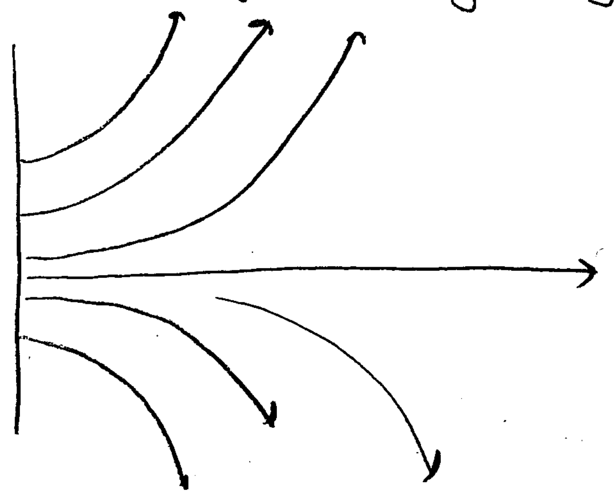
<u>Example</u>: (Exponential decay) $y' = -ky$ $(k > 0)$

$$\frac{dy}{dt} = -ky \Rightarrow \int \frac{dy}{y} = \int -k\,dt \Rightarrow \ln y = -kt + c$$

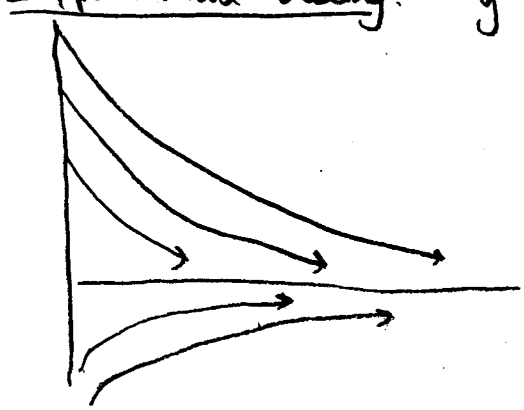
$$\Rightarrow y(t) = e^{-kt+c} = Ce^{-kt}$$

let's plot the solutions (say $k = \frac{1}{10}$)

<u>Exponential growth</u>: $y' = \frac{1}{10}y$, $y(t) = y_0 e^{\frac{1}{10}t}$



<u>Exponential decay</u>: $y' = -\frac{1}{10}y$, $y(t) = y_0 e^{-\frac{1}{10}t}$.

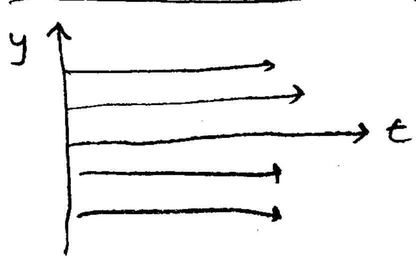


<u>Note</u>: $\lim_{t \to \infty} y(t) = 0$

<u>Question</u>: Can 2 of these solution curves ever intersect?

<u>Ans</u>: No (why?)

<u>What if $k=0$?</u> $(y'=0)$ $\frac{dy}{dt} = 0 \Rightarrow y(t) = 0$



Example (Decay to a limiting value)

$y' = k(72 - y)$

$\frac{dy}{dt} = -k(y-72)$

$\int \frac{dy}{y-72} = \int k\, dt$

$\ln|y-72| = kt + C$

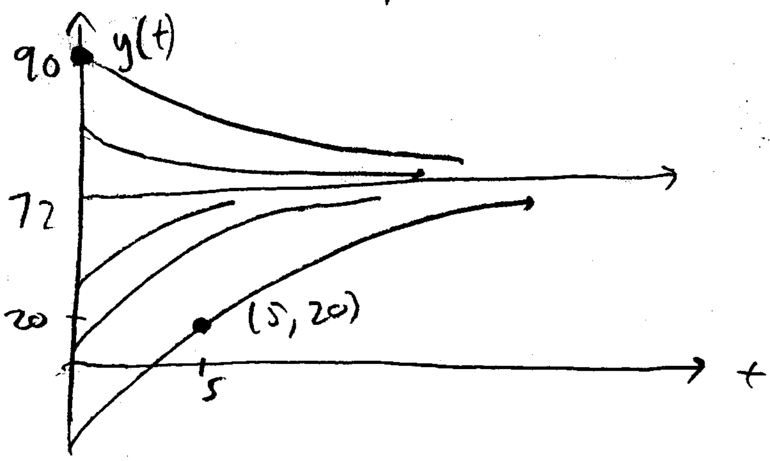
$y - 72 = e^{kt+C}$

$\boxed{y(t) = 72 + Ce^{-kt}}$

Question: What is C?

Ans: $y(0) = 72 + C$

"initial temp difference"




## Initial value problems (IVP's)

- Solving an ODE yields an infinite family of solutions.
- Once we specify a point $(t, y(t))$, we completely determine a particular solution.

Example: Consider the above example.

* Suppose $y(0) = 90$: $y(0) = 72 + C = 90 \Rightarrow C = 18$

  $y(t) = 72 + 18e^{-kt}$. This solution goes thru $(0, 90)$

* Suppose instead, $y(5) = 20$. This solution goes thru $(5, 20)$

## Solving word problems

Example (Exponential growth): A house sells in 2003 for \$ 179,500 & was on sale in 2008 for \$ 319,500.

(a) What was the rate of appreciation of the value?

$P'(t) = r\,P(t)$, Sol'n: $P(t) = C\,e^{rt}$

$P(0) = C = 179{,}500 \Rightarrow P(t) = 179{,}500 e^{rt}$.

$P(5) = 179{,}500 e^{5r} = 319{,}500$. (solve for $r$)

$e^{5r} = \frac{3195}{1795} \Rightarrow \ln(e^{5r}) = \ln\left(\frac{3195}{1795}\right)$

$\Rightarrow 5r = \ln\frac{3195}{1795} \Rightarrow r = \frac{1}{5}\ln\frac{3195}{1795}$

$\approx 11.5\%$.

(b) Suppose the market has been increasing at a 9% rate. How much is the house worth?

$r = \frac{9}{100}$, so $P(5) = 179{,}500\,e^{5\cdot\frac{9}{100}} = \$281{,}512$.

Example: (Exponential decay): You have 10 grams of a radioactive substance. 3 years later, you have 4 grams.

(a) What is the half-life?

(b) How long until only 1 gram remains?

$m'(t) = -k\,m(t)$ $k>0$, sol'n: $m(t) = Ce^{-kt}$.

$m(0) = C = 10 \Rightarrow \boxed{m(t) = 10e^{-kt}}$ · Need to find $k$.

(a) Half-life is amt of <u>time</u> until 5 grams remain,

$$m(t) = 10e^{-kt} = 5 \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow -kt = \ln\frac{1}{2}$$

solving for $t$: $t = -\frac{1}{k}\ln\frac{1}{2} = \boxed{\frac{1}{k}\ln 2}$. (we still need to solve for $k$).

<u>Note</u>: This does <u>not</u> depend on the init amt

Solve for $k$: $m(3) = 4 \Rightarrow m(3) = 10e^{-3k} = 4$.

$$\Rightarrow e^{-3k} = \frac{4}{10} \Rightarrow -3k = \ln\frac{2}{5}$$

$$\Rightarrow k = -\frac{1}{3}\ln\frac{2}{5} = \boxed{\frac{1}{3}\ln\frac{5}{2} = k}$$

Thus, half-life $= \frac{1}{k}\ln 2 = \frac{1}{1/3\ln 5/2}\cdot\ln 2 = \boxed{\frac{3\ln 2}{\ln 5/2}}$

(b) How long until 1 gram remains?

$$m(t) = 10e^{-kt} = 1 \quad \text{(solve for } t\text{)}$$

$$e^{-kt} = \frac{1}{10} \Rightarrow -kt = \ln\frac{1}{10} = -\ln 10$$

$$\Rightarrow t = \frac{1}{k}\ln 10 = \boxed{\frac{3\ln 10}{\ln 5/2}}$$

<u>Example</u> (Exponential decay to a value): My coffee is 120° when class starts, and the classroom is 75°. After 30 minutes, the coffee is 100°.

(a) What will the temp. be at the end of class (50 min)?

(b) Suppose it was brewed at 160°. When did I brew it?

$T'(t) = k(75 - T(t))$ $\quad k > 0.$ $\quad$ Sol'n: $T(t) = 75 + Ce^{-kt}$

$T(0) = 75 + C = 120 \Rightarrow C = 45$

$\boxed{T(t) = 75 + 45e^{-kt}}$ $\quad$ Need to find $k$

(a) $T(30) = 75 + 45e^{-30k} = 100 \Rightarrow 45e^{-30k} = 25$

$\Rightarrow e^{-30k} = \frac{25}{45} \Rightarrow -30k = \ln\frac{25}{45}$

$\Rightarrow k = -\frac{1}{30}\ln\frac{25}{45} = \boxed{30 \ln\frac{45}{25}}.$

$T(t) = 75 + 45e^{-30\ln\frac{45}{25}\cdot t}$

$\boxed{T(50) = 75 + 45e^{-\frac{3}{5}\ln 45/25}}$ (Temp at end of class)

(b) When was $T(t) = 160$?

$T(t) = 75 + 45e^{-kt} = 160 \Rightarrow 45e^{-kt} = 85$

$\Rightarrow e^{-kt} = \frac{85}{45} = \frac{17}{9} \Rightarrow -kt = \ln\frac{17}{9} \Rightarrow t = -\frac{1}{k}\ln\frac{17}{9}$

The coffee was brewed at $\boxed{t = -\dfrac{\ln 17/9}{30\ln 45/25}}$

Newton's 2nd law of motion: $\quad F = ma$

Gravitational acceleration: $a = -g = -9.8\ m/s^2$ (Think: why negative?)

Gravitational force: $F = ma = -mg$ (no air resistance)

Add air resistance: $F = -mg + R(v)$

grav. force

force due to air resistance. Function of velocity

Let's come up with a reasonable model for $R(v)$.

Facts of air resistance:
1. No velocity $\Rightarrow$ no air resistance
2. Acts in the opposite direction as velocity
3. Doesn't depend on $x$ (height).

Together, we conclude $R(v) = -\underbrace{r(v)}_{\text{non-negative, increasing.}}\, v$

It's not exact, but a good approximation is $r(v) = \overset{\text{const.}}{\overset{\downarrow}{r}}\, v$

i.e, $R(v) = -rv$ "air resistance force is $\approx$ proportional to velocity."

Therefore, $F = -mg + R(v) = -mg - rv$

Newton's $2^{nd}$ law: $F = ma = mv' = -mg - rv$

(Recall that $v'(t) = a(t)$) $\boxed{v' = -g - \frac{r}{m} v}$

Compare to decay $\to$ value equation: $T' = k(A - T) = kA - kT$.

Let's put it back into that form: $v' = -g - \frac{r}{m} v = \frac{r}{m}\left(-\frac{mg}{r} - v\right)$.

Here: $k = \frac{r}{m}$ and $A = -\frac{mg}{r}$.

The sol'n is thus $T(t) = A + Ce^{-kt}$

$$\boxed{v(t) = -\frac{mg}{r} + Ce^{-\frac{r}{m}t}}$$

__Note:__ Init. velocity $= v(0) = -\frac{mg}{r} + C$.

Terminal velocity $= \lim_{t\to\infty} v(t) = -\frac{mg}{r}$.

<u>Example:</u> A 70 kg object falls from rest, and its terminal velocity is $-20$ m/s.

(a) Find its velocity & distance traveled after 2 seconds.

(b) How long does it take to reach 80% of its terminal velocity?

$$v(t) = -\frac{mg}{r} + Ce^{-\frac{r}{m}t} = -20 + Ce^{-\frac{r}{70}t}$$

$$-\frac{mg}{r} = 20 = -\frac{70g}{r} \Rightarrow \boxed{r = \frac{7}{2}g}$$

$$\Rightarrow v(t) = -20 + Ce^{-\frac{1}{20}gt}$$

$$v(0) = -20 + C = 0 \Rightarrow C = 20$$

$$\Rightarrow \boxed{v(t) = -20 + 20e^{-\frac{1}{20}gt}}$$

(a) $\boxed{v(2) = -20 + 20e^{-g/10}}$. Recall that $x(t) = \int v(t)\,dt$.

$$\text{Dist} = \int_0^2 v(t)\,dt = \int_0^2 -20 + 20e^{-\frac{1}{20}gt}\,dt$$

$$= -20t\Big|_0^2 + \int_0^2 20e^{-\frac{1}{20}gt}\,dt = -40 + \frac{20e^{-\frac{1}{20}gt}}{-1/20\,g}\Bigg|_0^2$$

$$\boxed{\text{Dist} = -40 - \frac{400}{g}\left(e^{-g/10} - 1\right)}$$

(b)
$$v(t) = -20 + 20e^{-\frac{1}{20}gt} = \underbrace{-16}_{80\% \text{ of } -20}$$

$$20e^{-\frac{1}{20}gt} = 4 \Rightarrow e^{-\frac{1}{20}gt} = \frac{1}{5}$$

$$\Rightarrow -\frac{1}{20}gt = \ln\frac{1}{5} = -\ln 5$$

$$\Rightarrow \boxed{t = \frac{20\ln 5}{g}}$$