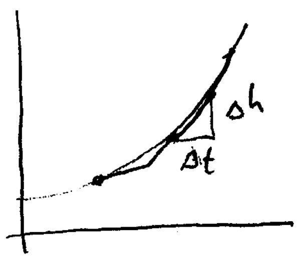
MTHSC 208, WEEK 3

Week 2 summary:

- Plotting solutions to autonomous ODE's (y' doesn't depend on t).
- Euler's method: $y' = f(t, y)$.



Given (t_0, y_0) & stepsize h (i.e, $y(t_0) = y_0$)

Method: $(t_{k+1}, y_{k+1}) = \left(t_k + \underbrace{h}_{\Delta t},\ y_k + \underbrace{h f(t_k, y_k)}_{\Delta y}\right)$.

- Solving ODEs by separation of variables.
- Difference between the general solution & a particular solution (with initial conditions).
- Another situation modeled by decay→value ODE: Falling objects with air resistance: $v' = \frac{r}{m}\left(\frac{-mg}{r} - v\right)$

Linear equations

Recall high school algebra:

A linear equation is $f(x) = ax + b$.

In Math 208:

A (1st order) linear differential equation is $y' = a(t)\,y + f(t)$ (wrt y, $a(t)$ & $f(t)$ are constants).

A (1st order) homogeneous linear diff. eq'n is $y' = a(t)\,y$.

Ex: $y' = t^2 y + 5$ linear

$y' = t y^2 + 5$ non-linear

$y' = t \sin y$ non-linear

$y' = (\sin t) y$ linear, homogeneous

$y' = t^3 - 2t^2 + t + 1$ linear

$y' = y \cdot y'$ non-linear.

We've seen how to solve homogeneous ODEs, say $y' = a(t) y$.

$$\frac{dy}{dt} = a(t)\, y \Rightarrow \int \frac{dy}{y} = \int a(t)\, dt \Rightarrow \ln|y| = \int a(t)\, dt + C$$

$$|y| = e^{\int a(t)\, dt + C} \Rightarrow |y| = e^{c} e^{\int a(t)\, dt} \Rightarrow y(t) = C e^{\int a(t)\, dt}$$

Now that we can solve $\underbrace{y'(t) = a(t)\, y(t)}_{\text{homogeneous eq'n}}$, let's solve $\underbrace{y'(t) = a(t)\, y(t) + f(t)}_{\text{inhomogeneous eq'n}}$

<u>Method #1</u>: Integrating factor ("product rule in reverse").

Write as $\boxed{y' - a(t)\, y(t) = f(t)}$

* Multiply both sides by $e^{-\int a(t)\, dt}$ "integrating factor"

$$e^{-\int a(t)\, dt} y' - a(t) e^{-\int a(t)\, dt} y = f(t) e^{-\int a(t)\, dt}$$

$$\left(y\, e^{-\int a(t)\, dt}\right)' = f(t) e^{-\int a(t)\, dt}$$ Now integrate both sides.

$$y(t)\,e^{-\int a(t)\,dt} = \int f(t)\,e^{-\int a(t)\,dt}\,dt$$ solving for $y(t)$ ...

$$\boxed{y(t) = e^{\int a(t)\,dt}\int f(t)\,e^{-\int a(t)\,dt}\,dt}$$

Example: $y' = 2y + t$. Think: why won't sep of vars. work?

$y' - 2y = t$ Integrating factor: e^{-2t}

$$y'e^{-2t} - 2ye^{-2t} = te^{-2t}$$

$$\int (ye^{-2t})' = \int te^{-2t}$$

$$ye^{-2t} = \frac{1}{4}e^{-2t}(2t+1) + C$$

$$\boxed{y(t) = -\frac{1}{2}t - \frac{1}{4} + Ce^{2t}}$$

Check: $y'(t) = \frac{1}{2} + 2Ce^{-2t} - 2$

$$-2y(t) = t + \frac{1}{2} + 2Ce^{2t}$$

$y' - 2y = t$ ✓ (i.e., $y(t)$ solves this ODE)

Let's practice getting the integrating factor

- $y' \boxed{+4} y = t^2$ int. factor e^{4t}, $\frac{d}{dt}e^{4t} = 4e^{4t}$

  $e^{4t}y' + 4e^{t}y = t^2e^{4t}$

  $(e^{4t}y)' = t^2e^{4t}$ now integrate & solve...

- $y' + (\sin t)y = 1$ int factor $e^{\cos t}$, $\frac{d}{dt}e^{\cos t} = \sin t\, e^{\cos t}$

  $e^{\cos t}y' + \sin t\, e^{\cos t}y = e^{\cos t}$

  $(e^{\cos t}y)' = e^{\cos t}$

- $y' - 12t^5 y = t^3$ int. factor e^{-2t^6}, $\frac{d}{dt} e^{-2t^6} = -12t^5 e^{-2t}$

$e^{-2t^6} y' - 12t^5 e^{-2t^6} y = e^{-2t^6} t^3$

$(e^{-2t^6} y)' = e^{-2t^6} t^3$

- $y' + \frac{1}{t} y = 1$ int. factor $e^{\ln t} = t$, $\frac{d}{dt} e^{\ln t} = \frac{d}{dt} t = 1$

$e^{\ln t} y' + \frac{1}{t} e^{\ln t} y = 1$

$t y' + y = 1$

$(t y)' = 1.$

<u>Method #2</u> (for solving 1st order linear ODE's): <u>Variation of parameters</u>

<u>Example</u>: $y' = 2y + t$

<u>Step 1</u>: Solve the "homogeneous part": $y_h' = 2y_h$

e.g., $y_h(t) = Ce^{2t}$

<u>Step 2</u>: Assume the gen'l sol'n is of the form $y(t) = v(t) y_h(t)$.

e.g., $y(t) = v(t) e^{2t}$.

<u>Step 3</u>: Plug this into the ODE & solve for $v(t)$

e.g., $y' = 2y + t$

$(ve^{2t})' = 2ve^{2t} + v'e^{2t}$

$\cancel{2ve^{2t}} + v'e^{2t} = \cancel{2ve^{2t}} + t$

$v'e^{2t} = t$

$v' = te^{-2t} \Rightarrow \int v'(t) = \int t e^{-2t}$

$\Rightarrow v(t) = -\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + C$

Step 4: Plug back into $y(t) = v(t)\, y_h(t)$

e.g., $y(t) = \left(-\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + C\right) e^{2t}$

$$\boxed{y(t) = -\tfrac{1}{2} t - \tfrac{1}{4} + C e^{2t}}$$

Both Integrating Factor & Variation of Parameters "work equally well."

Structure of the general sol'n of a 1^{st} order linear ODE:

$$\boxed{y' = ay + f}$$

Solution has the form $\boxed{y(t) = v(t)\, y_h(t)}$, where $y_h(t) = f(t)\, e^{-\int a(t)\,dt}$

It can be shown that $v'(t) = \dfrac{f(t)}{y_h(t)}$ (plug $y(t) = v(t)\, y_h(t)$ into $y' = ay + f$ & solve for $v(t)$; it's a bit messy)

Thus, $v(t) = \underbrace{\int f(t)\, e^{-\int a(t)\,dt}\, dt + C}$

plug this back into $y = v y_h$, to get...

$$(*) \quad y(t) = y_h(t) \int f(t)\, e^{-\int a(t)\,dt}\, dt + C\, y_h(t)$$

Now, let $y_p(t)$ be *any* particular sol'n to the ODE.

Then, $y_p(t)$ is of the form in $(*)$ for some C, let's say

$$(**) \quad y_p(t) = y_h(t) \int f(t)\, e^{-\int a(t)\,dt}\, dt + C_p\, y_h(t)$$

Now, subtract $(*) - (**)$

$\underbrace{y(t) - y_p(t)}_{\text{This is a sol'n to homog. eq'n.}} = (C - C_p)\, y_h(t)$. Let $A = C - C_p$.

Thus, $y(t) - y_p(t) = A\, y_h(t)$, i.e., $\boxed{y(t) = A\, y_h(t) + y_p(t)}$

* Big idea: The solution of a <u>linear</u> 1^{st} order ODE has the form

$y(t) = C y_h(t) + y_p(t)$ (or just $y(t) = y_h(t) + y_p(t)$, assuming that "$y_h(t)$ has C")

<u>Application</u>: Solve $T' = -k(T - A)$ (quickly!)

Homog. eq'n $T_h' = -k T_h$ has sol'n $T_h(t) = Ce^{-kt}$

Find any particular sol'n: $T_p(t) = A$ clearly works!

Thus, the general sol'n is $T(t) = T_h(t) + T_p(t) = A + Ce^{-kt}$.

* <u>Note</u>: If our ODE is autonomous, then there is <u>always</u> a const. sol'n!

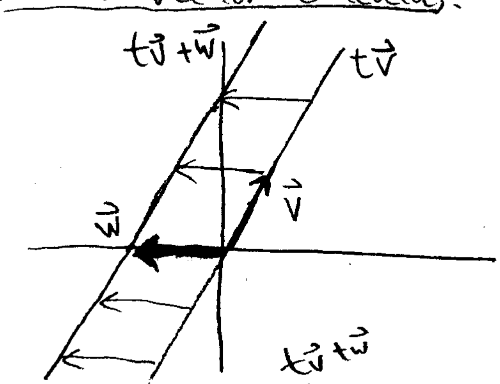
<u>Think</u>: What does this remind you of?

i.e., where have we seen this before?

$$y(t) = \underbrace{C}_{t}\ \underbrace{y_h(t)}_{\vec{v}} + \underbrace{y_p(t)}_{\vec{w}} \qquad \begin{array}{l} C \in (-\infty, \infty) \\ t \in (-\infty, \infty) \end{array}$$

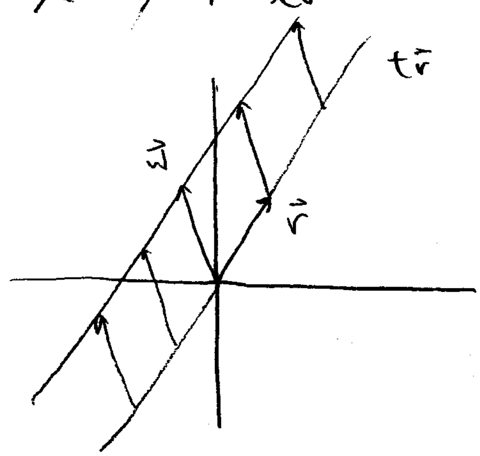
<u>Recall vector calculus</u>: $\ell = t\vec{v}$ is a line thru $\vec{0}$ ($y = mx$)

i.e, homogeneous.



<u>Q</u>: How do we parametrize a line not thru $\vec{0}$?

<u>A</u>: Add $\vec{w}$ to $t\vec{v}$, where $\vec{w}$ is <u>any</u> vector on the line.

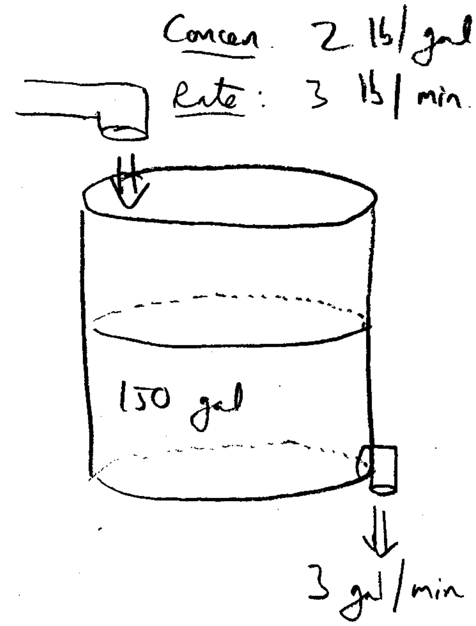


<u>Q</u>: Does "any" vector on ℓ work?

<u>A</u>: Yes!!! Any <u>particular</u> vector vector works, i.e, any (linear) line can be expressed as $C\vec{v}_h + \vec{v}_p$, $C \in (-\infty, \infty)$, i.e, $t\vec{v} + \vec{w}$, $t \in (-\infty, \infty)$.

Some more modeling applications with 1^{st} order ODE^s:

Mixing problems



Idea: Tank of fresh water.
Salt water flows in at some rate
Water drains at some rate.

Q: What is the concentration of salt at some time t?

Let $X(t) = $ # pounds of salt in the tank,

i.e, $\dfrac{X(t)}{\text{Vol}}$ = concentration of salt.

* Big idea: "$\underbrace{\text{rate of change of salt}}_{X'(t)}$ = rate in − rate out"

$$\begin{aligned}\text{rate in} &= (\text{volume rate})(\text{concentration}) \\ &= (3\ \text{gal/min})(2\ \text{lb/gal}) = 6\ \text{lb/min}\end{aligned}$$

$$\begin{aligned}\text{rate out} &= (\text{volume rate})(\text{concentration}) \\ &= (3\ \text{gal/min})\left(\frac{X(t)\ \text{lb}}{150\ \text{gal}}\right) = \frac{1}{50}X(t)\ \text{lb/min}\end{aligned}$$

Putting this together, $\boxed{X'(t) = 6 - \frac{1}{50}X(t)}$

Note: This problem had:
- one tank
- vol rate in = vol rate out

But these need not hold!

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Let's solve $\boxed{x' + \frac{1}{50}x = 6}$

Note: We could use:

(i) Separation of variables
(ii) integrating factor
(iii) variation of parameters
(iv) $y(t) = y_h(t) + y_p(t)$.

Let's use (iv), it's much easier!

Since our ODE is autonomous & linear, there's a const. sol'n:

$$x_p(t) = A \Rightarrow x_p'(t) = 0 \Rightarrow 0 + \frac{1}{50}A = 6 \Rightarrow A = 300$$

Also, the homog. eq'n $x_h' = -\frac{1}{50}x_h$ has sol'n $x_h(t) = Ce^{-\frac{1}{50}t}$

Thus, the general sol'n is $x(t) = x_h(t) + x_p(t)$

$$\boxed{x(t) = Ce^{-\frac{1}{50}t} + 300}$$

Recall: Tank initially contains fresh water: $x(0) = 0$

$$x(0) = 300 + C = 0 \Rightarrow C = -300$$

$$\Rightarrow \boxed{x(t) = 300 - 300e^{-\frac{1}{50}t}}$$

Note: $\lim_{t\to\infty} x(t) = 300$ i.e, the amount of salt approaches 300 lbs.

Does this make sense???

Check: As $t \to \infty$, concent $\longrightarrow 2$ lb/gal & vol $= 150$ gal. ✓

Note: Mathematically, this is the same as:

- Newton's law of heating/cooling
- Terminal velocity