MTHSC 208, WEEK 4

Week 3 summary

- Linear ODE's: $y'(t) = a(t)\,y(t) + f(t)$

  Homogeneous if $f(t) = 0$.

- 3 different ways to solve linear ODE's:

  (i) Integrating factor: $y' - ay = f$, int factor $= e^{-\int a(t)\,dt}$

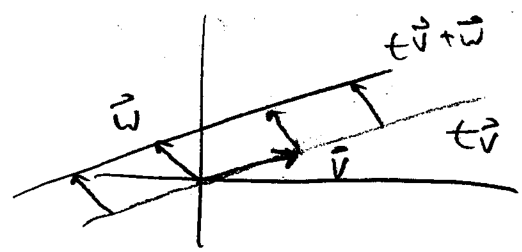
  "product rule in reverse"

  (ii) Variation of parameters: $y(t) = y_h(t)\,v(t)$. Solve for $v(t)$.

  (iii) Homogeneous + particular solution: $y(t) = y_h(t) + y_p(t)$

- Connection b/w general sol'n to linear ODE's & parametrized (linear!) lines.

  $y(t) = C\,y_h(t) + y_p(t)$ $\qquad c(t) = t\vec{v} + \vec{w}$



- Mixing problems: $x'(t) = (\text{rate in}) - (\text{rate out})$.

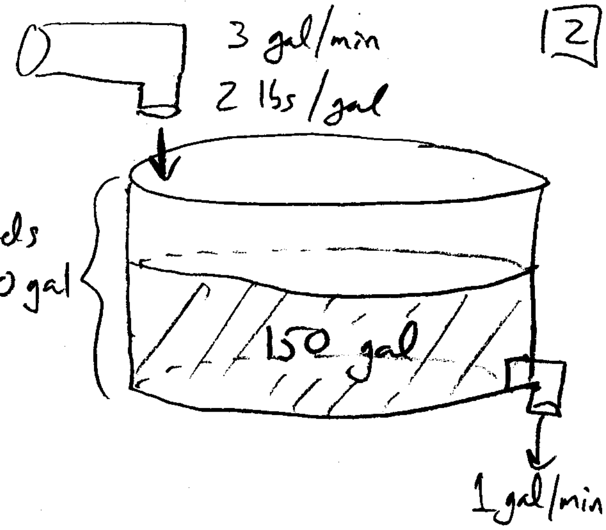
Mixing problems (cont.)

Let's consider a more complicated scenario.

Tank of fresh water

Salt water flows in at a faster rate than it drains.

Q: What is the concentration of salt at time t (before it overflows).



Again, let $x(t) = \#$ lbs salt in tank

Note: $\text{vol}(t) = 150 + 2t$, so it overflows at $t = 75$ min.

$$x'(t) = (\text{rate in}) - (\text{rate out})$$

$$\text{rate in} = (\text{vol. rate})(\text{concentration}) = \left(3 \tfrac{\text{gal}}{\text{min}}\right)\left(2 \tfrac{\text{lbs}}{\text{gal}}\right) = 6 \tfrac{\text{gal}}{\text{min}}$$

$$\text{rate out} = (\text{vol. rate})(\text{concentration}) = \left(1 \tfrac{\text{gal}}{\text{min}}\right)\left(\frac{x(t) \text{ lbs}}{150 + 2t \text{ gal}}\right) = \frac{1}{150+2t} x(t).$$

$$\boxed{x'(t) = 6 - \frac{1}{150+2t} x(t)}$$

Let's solve this: $x' + \frac{1}{150+2t} x = 6$ int. factor: $e^{\int \frac{1}{150+2t} dt}$

Note: $\int \frac{1}{150+2t} dt = \frac{1}{2} \int \frac{1}{75+t} dt = \frac{1}{2} \ln(75+t) + C$

Int. factor $= e^{\frac{1}{2}\ln(75+t)} = \left(e^{\ln 75+t}\right)^{1/2} = (75+t)^{1/2} = \sqrt{75+t}$

$$\left(x(75+t)^{1/2}\right)' = 6(75+t)^{1/2}$$

$$\Rightarrow x(75+t)^{1/2} = 6\int (75+t)^{1/2} dt = 4(75+t)^{3/2} + C$$

$$\Rightarrow x(t) = 4(75+t) + C(75+t)^{-1/2}$$

$$\Rightarrow \boxed{x(t) = 300 + 4t + \frac{C}{\sqrt{75+t}}} \quad \text{and} \quad x(0) = 0.$$

Plug in $x(0)=0$: $\quad x(0) = 300 + \dfrac{C}{\sqrt{75}} = 0 \Rightarrow C = -300\sqrt{75}$.

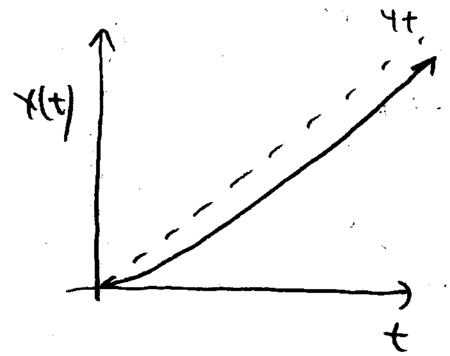
Particular sol'n: $\boxed{x(t) = 300 + 4t - \dfrac{300\sqrt{75}}{\sqrt{75+t}}}$

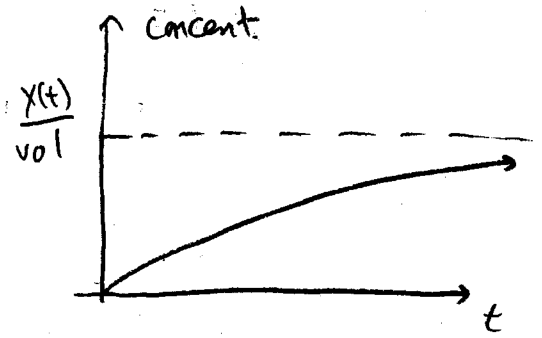
At $t = 75$, when the tank overflows, the salt content is

$$x(75) = 600 - \frac{300}{\sqrt{2}} \approx 387.87 \text{ lbs.}$$

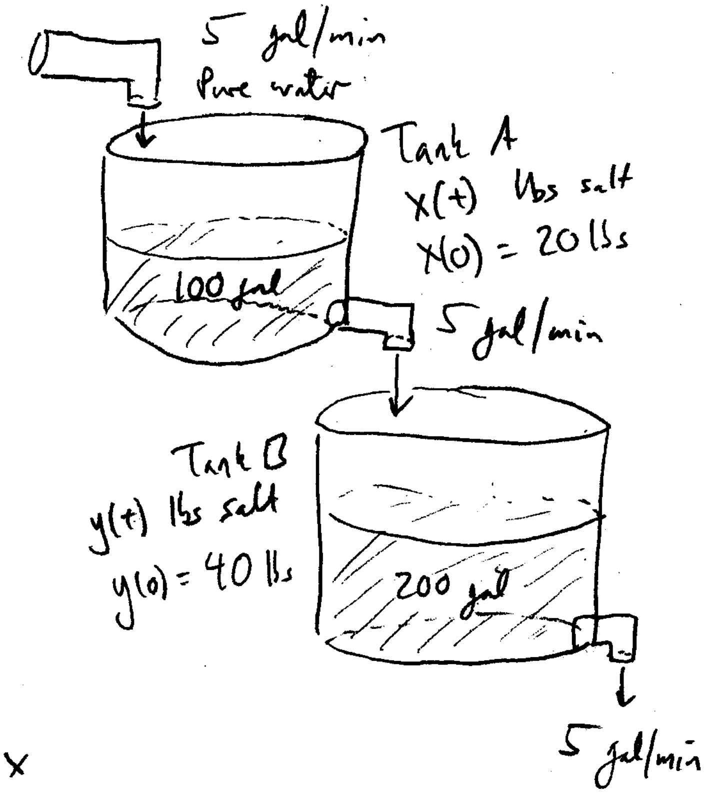
and concentration $= \dfrac{x(75)}{300} \approx \dfrac{387.87}{300} = 1.29$ lbs/gal

This makes sense





Mixing with 2 tanks



let $x(t)$ = amt of salt in tank A

let $y(t)$ = amt of salt in tank B.

Tank A: $x'(t) = (\text{rate in}) - (\text{rate out})$

rate in $= \left(5 \frac{\text{gal}}{\text{min}}\right)\left(0 \frac{\text{lbs}}{\text{gal}}\right)$

rate out $= \left(5 \frac{\text{gal}}{\text{min}}\right)\left(\frac{x(t) \text{ lbs}}{100 \text{ gal}}\right) = -\frac{1}{20}x$

Tank B: $y'(t) = (\text{rate in}) - (\text{rate out})$

rate in $= \left(5 \frac{\text{gal}}{\text{min}}\right)\left(\frac{x(t) \text{ lbs}}{100 \text{ gal}}\right) = \frac{1}{20}x$

rate out $= \left(5 \frac{\text{gal}}{\text{min}}\right)\left(\frac{y(t) \text{ lbs}}{200 \text{ gal}}\right) = \frac{1}{40}y$

We get a system of ODEs: $\boxed{\begin{array}{ll} x' = -\frac{1}{20}x & x(0)=20 \\ y' = \frac{1}{20}x - \frac{1}{40}y & y(0)=40 \end{array}}$

$$x(t) = 20e^{-\frac{1}{20}t}$$

Plug this in: $y' = \frac{1}{20}\left(20e^{-\frac{1}{20}t}\right) - \frac{1}{40}y$

$$y' + \frac{1}{40}y = e^{-\frac{1}{20}t} \qquad \text{int. Factor} = e^{\frac{1}{40}t}$$

$$\left(ye^{\frac{1}{40}t}\right)' = e^{-\frac{1}{20}t}\cdot e^{\frac{1}{40}t} = e^{-\frac{1}{40}t}$$

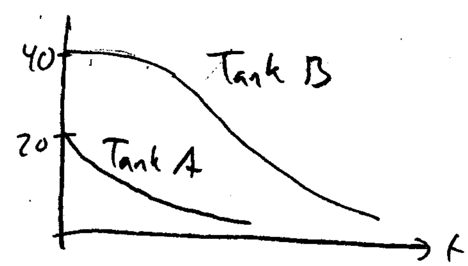
$$ye^{\frac{1}{40}t} = \int e^{-\frac{1}{40}t}\,dt = -40e^{-\frac{1}{40}t} + C$$

$$y(t) = \left(-40e^{-\frac{1}{40}t} + C\right)e^{-\frac{1}{40}t} = -40e^{-\frac{1}{20}t} + Ce^{-\frac{1}{40}t}$$

$$y(0) = -40 + C = 40 \Rightarrow C = 80$$

$$\boxed{y(t) = -40e^{-\frac{1}{20}t} + 80e^{-\frac{1}{40}t}}$$

Let's plot $x(t)$ & $y(t)$:



Logistic equation

Recall exponential growth: $y'(t) = r\,y(t)$.
rate r doesn't depend on $y(t)$.

Suppose $y(t)$ = population of a colony.

When population is small, it grows $\approx$ exponentially
When population $\approx$ carrying capacity, it grows slowly.
When population > capacity, it decreases.

In general, the "rate" r decreases as y increases.

How do we model this?

We want $y'(t) = r(y)\, y(t)$ where $r(y)$ is decreasing.

Try $r(y) = r - ay$, where $a > 0$ is fixed

Check: when $y = 0$, $r(y) = r$ ✓

when $y = \frac{r}{a}$, $r(y) = 0$ ✓

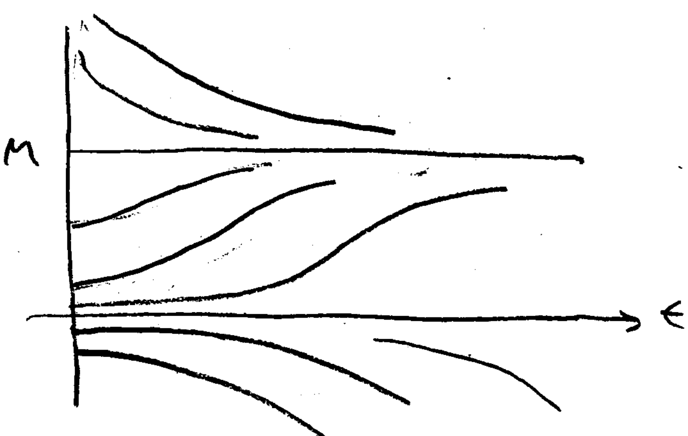
when $y > \frac{r}{a}$, $r(y) < 0$ ✓

This threshold $M := \frac{r}{a}$ represents the "carrying capacity," $a = \frac{r}{M}$)

We have $y'(t) = r(y)\, y(t) = (r - ay) y = (r - \frac{r}{M} y) y = r(1 - \frac{y}{M}) y$.

* The equation $\boxed{y' = r\left(1 - \frac{y}{M}\right) y}$ is the <u>logistic equation</u>.

Note: It is autonomous:
The steady-state sol'ns are $y(t) = 0$, $y(t) = M$.



This can be solved using separation of variables (it's messy).

The sol'n is $\boxed{y(t) = \dfrac{M}{1 + Ce^{-rt}}}$

Note: Initial population is $y(0) = \dfrac{M}{1+C}$

Limiting population is $\lim_{t \to \infty} y(t) = M$.

Example: The mass $m(t)$ of a colony of bacteria satisfies the logistic equation. The petrie dish holds 50 grams. Initially, there are 10 grams, & mass is increasing at 1 gram/day. Find $m(t)$.

We know $m'(t) = r\left(1 - \frac{m(t)}{50}\right) m(t)$

$$m(t) = \frac{50}{1+Ce^{-rt}}, \quad m(0) = 10 \text{ gram}$$

$$m'(0) = 1 \text{ gram/day.}$$

Note: This does not imply that $r = 1$.

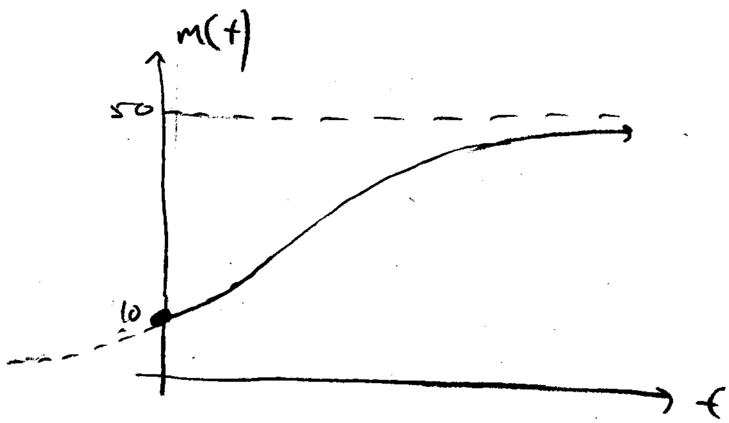
(Compare 1% interest rate, vs. earning $y'(0) = \$1/\text{day}$).

$$m(0) = \frac{50}{1+C} = 10 \Rightarrow C = 4 \Rightarrow m(t) = \frac{50}{1+4e^{-rt}}.$$

$$m'(0) = r\left(1 - \frac{m(0)}{50}\right) m(0)$$

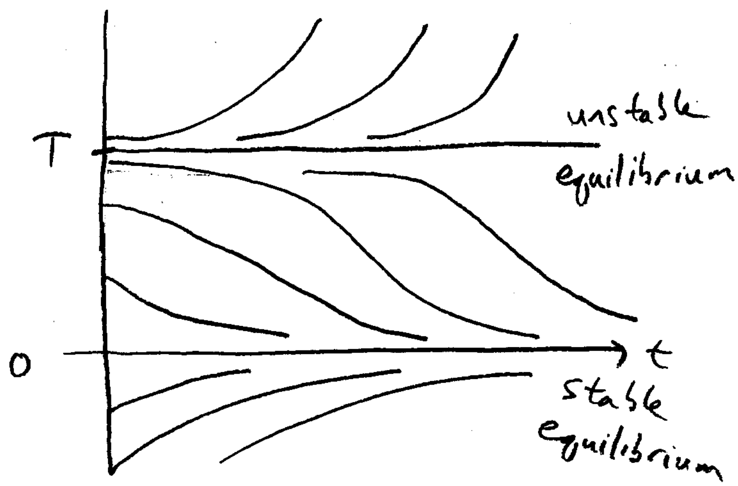
$$1 = r\left(1 - \frac{10}{50}\right) \cdot 10 = r \cdot \frac{4}{5} \cdot 10 = 1 \Rightarrow r = \frac{1}{8}.$$

The particular sol'n is thus $\boxed{m(t) = \frac{50}{1+4e^{-t/8}}}$



<u>Question:</u> What if we replace r with $-r$ in the logistic equation?

We'd get an ODE:



$$\boxed{y' = -r\left(1 - \frac{y}{T}\right) y}$$

The solution would still be

$$y(t) = \frac{T}{1 + Ce^{rt}}.$$

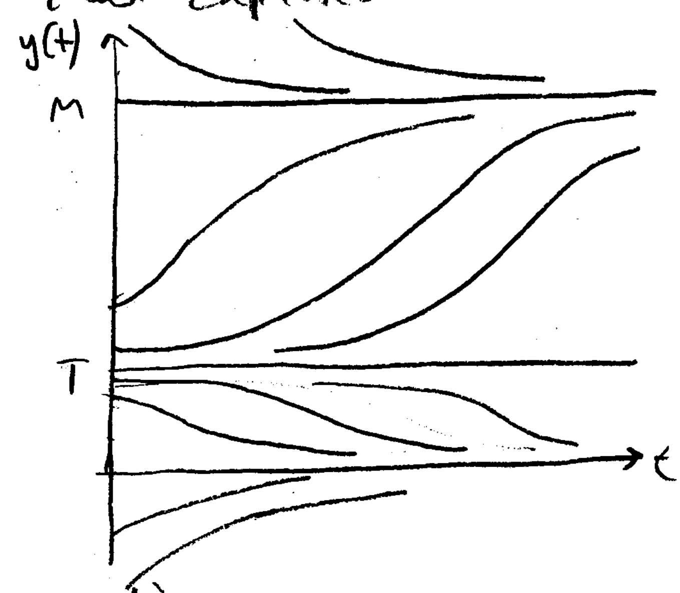
But, $Ce^{rt} \longrightarrow \infty$ as $t \to \infty$ (Actually, in finite time!)

This models a population with a "threshold" T, i.e.,

if $y(t) < T$, then the population dies out, but

if $y(t) > T$, then the population "explodes."

Realistically, we'd like a model that captures both phenomena



Let's "make" an ODE have steady-state solutions $y(t) = M, T, \; \& \; 0$.

$$\boxed{y'(t) = -r\left(1 - \frac{y}{M}\right)\left(1 - \frac{y}{T}\right) y}$$

This actually modeled the (now extinct) passenger pigeon quite accurately.

2nd order ODE's

We will consider equations of the form $y'' = f(t, y, y')$.

A solution is any function $y(t)$ s.t. $y''(t) = f(t, y(t), y'(t))$.

Motivating example: $F = ma$ (Newton's 2nd law of motion).

Force (could be gravitational, mechanical, etc) can be a function of time, displacement $x(t)$, and velocity $x'(t)$.

$$F(t, x, x') = m x''(t).$$

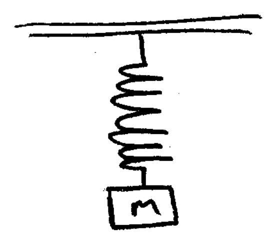
Ex 1: Gravity ("constant" force): $\boxed{m x''(t) = -mg}$

Ex 2: Spring  (at rest).

Hooke's law: Restoring force $R(x) = -kx \Rightarrow \boxed{m x''(t) = -kx}$

Think: "Force is proportional to how much we stretch or compress."

Ex 3: Now, suppose the weight is hanging:



Forces add, so $F = R(x) + (\text{Grav. force})$

$$\boxed{m x'' = -kx + mg}$$ (Note: Why $+mg$?)

Ex 4: Suppose there's also a damping force (springs never "bounce forever").

This is like air resistance:
- Proportional to velocity
- Acts against the direction of motion.

Thus, $D(x') = -\mu x'$, μ is const.

Forces add, so $F = D(x') + R(x) + mg \Rightarrow \boxed{m x'' = -\mu x' - kx + mg}$

There are 2 "general techniques" for analyzing 2^{nd} order ODEs:

(i) Solving them directly

(ii) Turning them into systems of 1^{st} order ODEs

<u>example</u>: $y'' + 3ty' + 2y = \sin t$.

let $v = y'$, so $v' = y''$

we now have: $\begin{cases} v' + 3tv + 2y = \sin t \\ v = y' \end{cases}$

We will do (i) first, because it's an extension of what we've done for 1^{st} order systems.

A <u>linear</u> 2^{nd} order ODE has the form $y'' + p(t)y' + q(t)y = g(t)$.

A <u>homogeneous</u> (linear) 2^{nd} order ODE is $y'' + p(t)y' + q(t)y = 0$

* <u>Big idea</u>: The general solution to a linear 2^{nd} order ODE is $\boxed{y(t) = \underbrace{C_1 y_1(t) + C_2 y_2(t)}_{y_h(t)} + y_p(t)}$,

where $y_p(t)$ is <u>any</u> particular sol'n.

<u>Take-home message</u>: There is a 2-parameter infinite family of sol'ns,

Examples:

- Find the general sol'n to $\boxed{y'' = k^2 y}$.

  Observe that $y_1(t) = e^{kt}$ works, as does $y_2(t) = e^{-kt}$.

  Thus, the general sol'n is $\boxed{y(t) = C_1 e^{kt} + C_2 e^{-kt}}$

- Find the general sol'n to $\boxed{y'' = -k^2 y}$

  Observe that $y_1(t) = \cos kt$ works; as does $y_2(t) = \sin kt$.

  Thus, the general sol'n is $\boxed{y(t) = A \cos kt + B \sin kt}$

- Find the general sol'n to $\boxed{y'' - 3y' + 2y = 0}$

  What might be a good guess?

  Try $y(t) = e^{rt}$ where r is some const.

  Solve for r:
  $$y = e^{rt}$$
  $$y' = re^{rt}$$
  $$y'' = r^2 e^{rt}$$

  Plug back into $y'' - 3y' + 2y = 0$
  $$r^2 e^{rt} - 3re^{rt} + 2e^{rt} = 0$$
  $$e^{rt}(r^2 - 3r - 2) = 0$$
  $$e^{rt}(r-1)(r-2) = 0 \Rightarrow r = 1 \text{ or } 2.$$

  Thus, we've found two sol'ns: $y_1(t) = e^t$, $y_2(t) = e^{2t}$,

  so the general sol'n is $\boxed{y(t) = C_1 e^t + C_2 e^{2t}}$

<u>Question</u>: what if we have a repeated root?

e.g., $\boxed{y'' - 6y' + 9y = 0}$

Again, guess $\left.\begin{array}{l} y = e^{rt} \\ y' = re^{rt} \\ y'' = r^2 e^{rt} \end{array}\right\}$ $\begin{array}{l} r^2 e^{rt} - 6re^{rt} + 9e^{rt} = 0 \\ e^{rt}(r^2 - 6r + 9) = 0 \\ (r-3)^2 = 0 \quad \Rightarrow r = 3 \end{array}$

We've determined that $y_1(t) = C_1 e^{3t}$ is a solution.

But we need one more!

Try $y(t) = v(t)e^{3t}$, and solve for $v(t)$.

If $y = ve^{3t}$, then $y' = 3e^{3t}v + e^{3t}v'$, and

$$y'' = 3(3e^{3t}v + e^{3t}v') + (3e^{3t}v' + e^{3t}v'')$$
$$= 9e^{3t}v + 6e^{3t}v' + e^{3t}v''$$

Plug back into ODE:

$$\underbrace{(\cancel{9e^{3t}v} + \cancel{6e^{3t}v'} + e^{3t}v'')}_{y''} - 6\underbrace{(\cancel{3e^{3t}v} + \cancel{e^{3t}v'})}_{y'} + 9\underbrace{(\cancel{e^{3t}v})}_{y} = 0$$

$$v''e^{3t} = 0 \quad \Rightarrow \quad v'' = 0 \quad \Rightarrow \quad v(t) = Ct + D$$

<u>Conclusion</u>: e^{3t} is a sol'n, <u>and</u> $(Ct + D)$ is a sol'n for <u>any</u> C & D.

Since any C & D will do, let's choose $C = 1$, $D = 0$,

so $v(t) = t$.

Now, $y_1(t) = e^{3t}$, $y_2(t) = v(t)e^{3t} = te^{3t}$, so the general solution is $\boxed{y(t) = C_1 e^{3t} + C_2 t e^{3t}}$