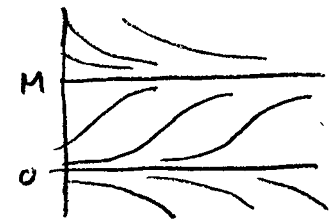
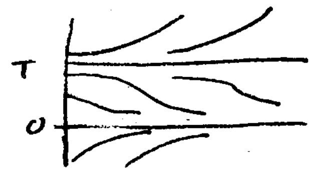
MTHSC 208, WEEK 5

Week 4 summary

- Logistic equation : $y' = r\left(1-\frac{y}{M}\right)y, \quad y(t) = \frac{M}{1+Ce^{-rt}}$

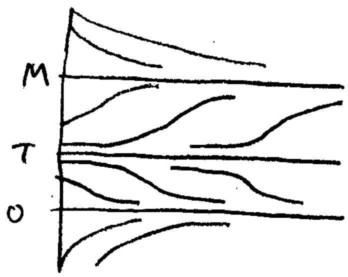


"Threshold" equation : $y' = -r\left(1-\frac{y}{T}\right)y, \quad y(t) = \frac{T}{1+Ce^{rt}}$



Put these together:

$$y' = -r\left(1-\frac{y}{M}\right)\left(1-\frac{y}{T}\right)y$$



- 2nd order linear ODEs : $y'' + p(t)y' + q(t)y = f(t)$

General solution: $y(t) = y_h(t) + y_p(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$.

- Constant coefficients : $y'' + py' + qy = 0$

Assume the sol'n has the form: $y(t) = e^{rt}$.

Plug back in ; solve for r. Get $e^{rt}(r^2+pr+q) = 0$

3 cases: (i) $r_1 \neq r_2$ real. $\quad y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

(ii) $r_1 = r_2 \quad y(t) = C_1 e^{r_1 t} + C_2 t e^{r_2 t}$

(iii) $r_{1,2} = a \pm bi \quad$ (Preview): $y(t) = e^{at}(A\cos bt + B\sin bt)$.

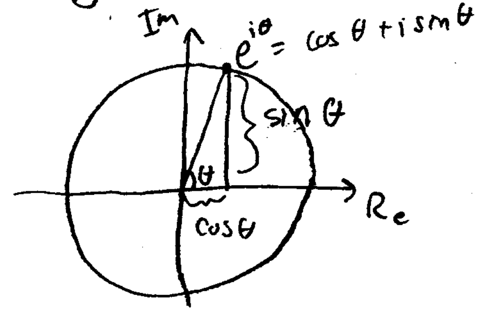
Let's consider Case (iii) : $y'' + py' + qy = 0$ and the roots are $r_{1,2} = a \pm bi$.

We have 2 solutions, $y_1(t) = e^{(a+bi)t}$, $\quad y_2(t) = e^{(a-bi)t}$

Thus, the general solution is $y(t) = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t}$.

What does these functions look like???

There's indeed a "better way" to write this general solution.



* Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Recall: $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$

$$y_1(t) = e^{(a+bi)t} = e^{at} e^{ibt} = e^{at}(\cos bt + i \sin bt)$$

$$y_2(t) = e^{(a-bi)t} = e^{at} e^{ibt} = e^{at}(\cos(-bt) + i\sin(-bt))$$

$$= e^{at}(\cos bt - i \sin bt)$$

Recall: Since our ODE is linear & homogeneous, we can

- add two solutions
- multiply two solutions

... and still have a solution.

Thus, $\frac{1}{2}(y_1(t) + y_2(t)) = e^{at}\cos bt$ is a solution

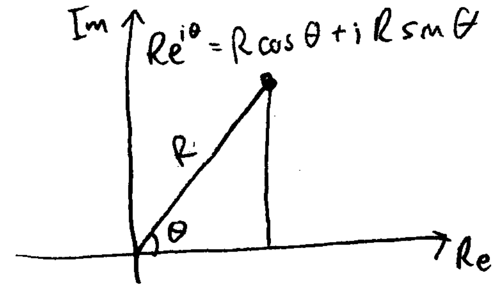
and $\frac{1}{2i}(y_1(t) - y_2(t)) = e^{at}\sin bt$ is a solution

Conclusion: The general solution can be written as

$$y(t) = Ae^{at}\cos bt + Be^{at}\sin bt, \quad \text{or} \quad \boxed{y(t) = e^{at}(A\cos bt + B\sin bt)}$$

<u>Review:</u> Basic complex numbers & Euler's formula

In the complex plane, a point z at dist R from $\vec{0}$ & angle θ is



$$Re^{i\theta} = R\cos\theta + iR\sin\theta.$$

From this, it is "easy" to see that

$$\cos(-\theta) = \cos\theta \quad \text{and} \quad \sin(-\theta) = -\sin\theta.$$

($\theta \mapsto -\theta$ is a reflection across the x-axis. This preserves the x-coordinate but flips the sign of the y-coordinate)

Euler's formula is $\boxed{e^{it} = \cos t + i\sin t}$, and it implies some neat facts:

$$\left.\begin{aligned} e^{it} &= \cos t + i\sin t \\ e^{-it} &= \cos t - i\sin t \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \tfrac{1}{2}\left(e^{it} + e^{-it}\right) &= \cos t \\ \tfrac{1}{2i}\left(e^{it} - e^{-it}\right) &= \sin t \end{aligned}}$$

Also, $\boxed{e^{i\pi} = -1}$! Visually,

$\theta = \pi$ radians

Another way to see Euler's formula, from Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \frac{(it)^6}{6!} + \frac{(it)^7}{7!} + \frac{(it)^8}{8!} + \cdots$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} +/- \cdots$$

$$\sin t = it - i\frac{t^3}{3!} + i\frac{t^5}{5!} - i\frac{t^7}{7!} +/- \cdots$$

$$\Rightarrow e^{it} = \cos t + i\sin t.$$

Inhomogeneous linear 2^{nd} order ODE's:

"Forcing term"

$$y'' + py' + qy = f, \qquad p(t),\ q(t),\ f(t)$$

* Big idea: $y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$, where y_1, y_2 are sol'ns to the homogeneous equation, and $y_p(t)$ is <u>any</u> solution to the (inhomogeneous) ODE.

Proof: Suppose we have a particular sol'n $y_p(t)$.

Let $y(t)$ be <u>any</u> other solution. Then

$$\begin{array}{l} y'' + py' + qy = f \\ -(y_p'' + py_p' + qy_p = f) \\ \hline (y-y_p)'' + p\cdot(y-y_p)' + q\cdot(y-y_p) = 0 \end{array}$$

Thus, $y - y_p$ is a solution to the homogeneous equation,

i.e., $y - y_p = C_1 y_1 + C_2 y_2 \Rightarrow y = C_1 y_1 + C_2 y_2 + y_p$.

Method of undetermined coefficients:

- Works only when coefficients p & q are constants.

* Big idea: Usually we can <u>guess</u> the form of a particular sol'n!

Example 1: $y'' - 5y' + 4y = e^{3t}$

First, solve the homog. eq'n: $y_h(t) = C_1 e^{4t} + C_2 e^{t}$.

Next, <u>guess</u> that $y_p(t) = ae^{3t}$ (Think: Why will this work??)

$$y_p'(t) = 3ae^{3t}, \qquad y_p''(t) = 9ae^{3t}$$

Plug back in & solve for a:

$(9y'' \quad - \quad 5 \quad y' \quad + \quad 4 \quad y \quad = e^{3t})$

$$9ae^{3t} - 5(3ae^{3t}) + 4ae^{3t} = e^{3t}$$

Combine terms... $\quad -2ae^{3t} = e^{3t} \Rightarrow a = -\frac{1}{2}$

Thus, $y_p(t) = -\frac{1}{2}e^{3t}$ and $y(t) = y_h(t) + y_p(t)$, so

$$\boxed{y(t) = C_1 e^{4t} + C_2 e^{t} - \frac{1}{2}e^{3t}}$$

Think: Why did this work?

* Because the forcing term $f(t)$ and its derivatives had the same form (up to constants).

Example 2: $\quad y'' + 2y' - 3y = 5\sin 3t. \qquad y_h(t) = C_1 e^{t} + C_2 e^{-3t}$

Problem: If $y_p(t) = a\sin 3t$, $y_p'(t) = 3a\cos 3t$.

Not of the "same form" (so they won't cancel).

How do we fix this?

Ans: Consider a more general particular sol'n:

$$\left.\begin{aligned} y_p(t) &= A\cos wt + B\sin wt \\ y_p'(t) &= -wA\cos wt + wB\sin wt \end{aligned}\right\} \text{ have the same form!}$$

So, assume that there's a solution of the form

$$\boxed{y_p(t) = a\cos 3t + b\sin 3t}$$

$$y_p'(t) = -3a\sin 3t + 3b\cos 3t$$

$$y_p''(t) = -9a\cos 3t - 9b\sin 3t.$$

Plug back in:

$$y_p'' + 2y_p' - 3y_p = (-9a\cos 3t - 9b\sin 3t) + (-6a\sin 3t + 6b\cos 3t) - (3a\cos 3t + 3b\sin 3t)$$

$$= \underbrace{(-12a + 6b)}_{=0}\cos 3t + \underbrace{(-6a - 12b)}_{=5}\sin 3t = \underbrace{5\sin 3t}_{F(t)}$$

Thus, we have $\left.\begin{array}{l} -12a + 6b = 0 \\ -6a - 12b = 5 \end{array}\right\} \Rightarrow a = -\frac{1}{6},\quad b = -\frac{1}{3}$

$$\Rightarrow y_p(t) = -\tfrac{1}{6}\cos 3t - \tfrac{1}{3}\sin 3t$$

The general solution is therefore

$$y(t) = y_h(t) + y_p(t) = \boxed{C_1 e^{t} + C_2 e^{-3t} + \left(-\tfrac{1}{6}\cos 3t - \tfrac{1}{3}\sin 3t\right)}.$$

Example 3: (polynomial forcing term).

$$y'' + 2y' - 3y = 6t^2 + t - 2.$$ Again, $y_h(t) = C_1 e^{t} + C_2 e^{-3t}$

Assume there's a particular sol'n of the form $\boxed{y_p(t) = at^2 + bt + c}$

Why?? (b/c $y_p'' + 2y_p' - 3y_p$ will also be a degree-2 poly.)

So, all we have to do is find a, b, c.

Plug back in: $\quad y_p' = 2at + b, \quad y_p'' = 2a$

$$y'' + 2y' - 3y = (2a) + 2(2at + b) - 3(at^2 + bt + c) = 6t^2 + t - 2$$

$$\underbrace{(-3a)}_{=6}t^2 + \underbrace{(4a - 3b)}_{=1}t + \underbrace{(2a + 2b - 3c)}_{=-2} = 6t^2 + t - 2$$

$$\begin{cases} -3a = 6 \\ 4a - 3b = 1 \\ 2a + 2b - 3c = -2 \end{cases} \Rightarrow \begin{array}{l} a = -2 \\ b = -3 \\ c = -\frac{8}{3} \end{array}$$

$$\Rightarrow y_p(t) = -2t^2 - 3t - \tfrac{8}{3}$$

$$\Rightarrow y(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = C_1 e^{t} + C_2 e^{-3t} - 2t^2 - 3t - \tfrac{8}{3}}$$

What could go wrong with this method?

What if the forcing term is a solution to the homogeneous eq'n?

Example 4: $y'' - 3y' + 2y = e^{2t}$.

Characteristic eq'n: $r^2 - 3r + 2 = (r-1)(r-2)$

$\Rightarrow \; y_h(t) = C_1 e^{t} + C_2 e^{2t}$

Assume there's a particular sol'n of the form $y_p(t) = ae^{2t}$.

$y_p' = 2ae^{2t}, \quad y_p'' = 4ae^{2t}$.

Plug back in: $(4ae^{2t}) - 3(2ae^{2t}) + 2(ae^{2t}) = e^{2t}$

$\Rightarrow 0ae^{2t} = e^{2t}$ No solution for a!

What happened?

Note: ae^{2t} solves: $y'' - 3y' + 2y = 0$ (homog. eq'n),

thus it will never solve $y'' - 3y' + 2y = e^{2t} \neq 0$.

To "fix" this, assume instead that $\boxed{y_p(t) = ate^{2t}}$

$y_p'(t) = 2ate^{2t} + ae^{2t}, \qquad y_p''(t) = 4ate^{2t} + 4ae^{2t}$

Plug back in: $(4ate^{2t} + 4ae^{2t}) - 3(2ate^{2t} + ae^{2t}) + 2(ate^{2t}) = e^{2t}$

$\Rightarrow 0ate^{2t} + ae^{2t} = e^{2t}$

$\Rightarrow a = 1 \quad \Rightarrow \quad y_p(t) = te^{2t}$

Thus, the general sol'n is $y(t) = y_h(t) + y_p(t)$

$$\boxed{y(t) = C_1 e^{t} + C_2 e^{2t} + te^{2t}}$$

Example 5: $y'' + 2y' - 3y = 5\sin 3t + 6t^2 + t - 2 \quad (*)$

Again, $y_h(t) = C_1 e^t + C_2 e^{-3t}$.

Recall that $-\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$ solves $y'' + 2y' - 3y = 5\sin 3t$ (Ex 2)

and $-2t^2 - 3t - \frac{8}{3}$ solves $y'' + 2y' - 3y = 6t^2 + t - 2$ (Ex 3)

Convince yourself that $y_p(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t - 2t^2 - 3t - \frac{8}{3}$ solves $(*)$.

Thus, the gen'l solution is $y(t) = y_h(t) + y_p(t)$, i.e,

$$\boxed{y(t) = C_1 e^t + C_2 e^{-3t} - \frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t - 2t^2 - 3t - \frac{8}{3}}$$

In general: (Combination forcing terms).

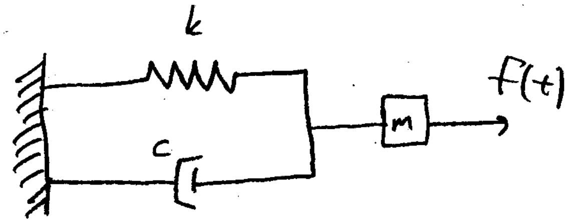
$*$ Suppose $y'' + py' + qy = f(t)$ has sol'n $y_f(t)$

and $y'' + py' + qy = g(t)$ has sol'n $y_g(t)$,

then $y'' + py' + qy = \alpha f(t) + \beta g(t)$ has sol'n $\alpha y_f(t) + \beta y_g(t)$.

Application: Harmonic Motion.

Recall mass-spring systems:



Let $x(t)$ = displacement of the mass.

Then $x(t)$ satisfies the following ODE: $mx'' + 2cx' + \omega_0^2 x = f(t)$

(c = damping const, ω_0 = frequency, $f(t)$ driving force)

Most basic case: Simple harmonic motion (no damping or driving force).

$x'' + kx = 0 \qquad k = \omega^2 > 0 \qquad$ (Here, ω will be "frequency").

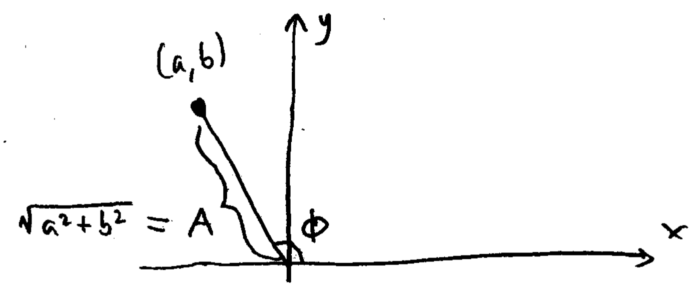
$\boxed{x'' = -\omega^2 x} \Rightarrow \boxed{x(t) = a\cos\omega t + b\sin\omega t}$

(what does this function "look like"?)

Here's a trick: We can actually write it as a single cosine wave!

Let's switch to polar coordinates:

$* \; (a,b) = (A\cos\phi, A\sin\phi)$



$A \geq 0, \quad -\pi < \phi \leq \pi$

Sneaky little trick:

$$x(t) = \underbrace{a}\cos(\omega t) + \underbrace{b}\sin(\omega t)$$

$$= A\cos\phi\cos(\omega t) + A\sin\phi\sin(\omega t)$$

$$= A\cos(\phi - \omega_0 t) \qquad \text{Trig identity: } \cos(x-y) = \cos x\cos y + \sin x \sin y$$

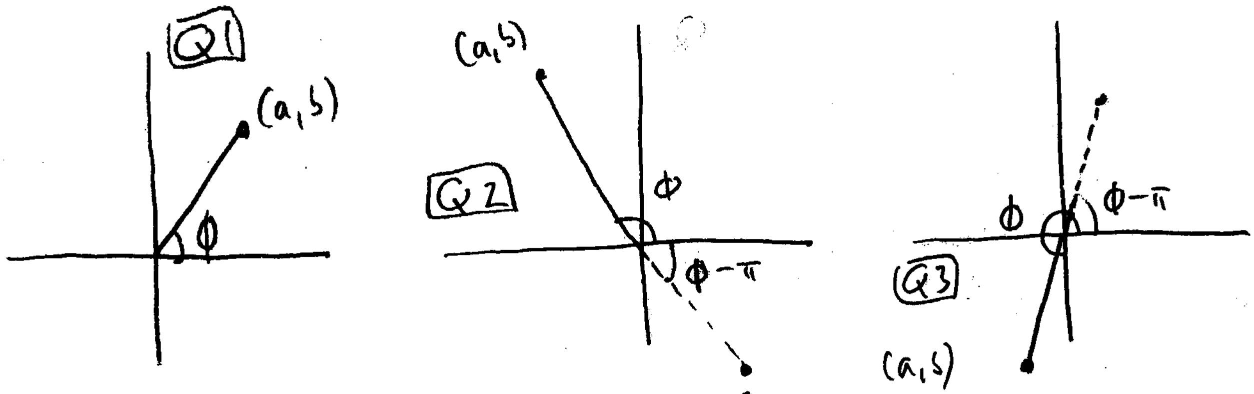
$$= A\cos(\omega t - \phi).$$

Big idea: Any function $x(t) = a\cos(\omega t) + b\sin(\omega t)$ can be written as a single cosine wave, with:

* amplidude $A = \sqrt{a^2+b^2}$
* phase shift $\frac{\phi}{\omega}$, where "$\phi = \tan^{-1}(b/a)$.

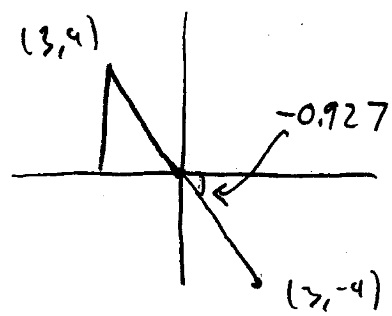
So, $x(t) = A\cos(\omega t - \phi) = \boxed{A\cos\left(\omega\left(t - \frac{\phi}{\omega}\right)\right)}$

Note: Since $\overbrace{-\frac{\pi}{2} < \phi < \frac{\pi}{2}}^{\text{for calculators}}$, $\phi = \begin{cases} \arctan(b/a) & Q\ 1,4 \\ \arctan(b/a) + \pi & Q\ 2 \\ \arctan(b/a) - \pi & Q\ 3 \end{cases}$



what your calculator would say (a,b) is.

Example: $x(t) = -3\cos t + 4\sin t$



$A = \sqrt{3^2+4^2} = 5$

$\arctan(-4/3) = -0.927$

according to your calculator

So $\phi = -0.927 + \pi$

$\Rightarrow x(t) = 5\cos[t - (-0.927 + \pi)]$