# MTHSC 208, WEEK 6

**Week 5 summary:**

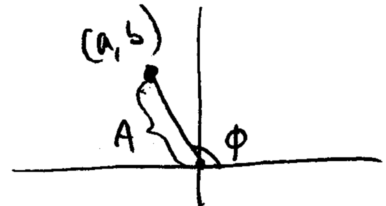
- **Method of undetermined coefficients.**
  Used to solve linear inhomogeneous $2^{nd}$ order ODE$^s$ with constant coefficients.

  **Big idea:** Guess $y_p(t)$ to have the same form as the forcing term $f(t)$. Then, $y(t) = y_h(t) + y_p(t)$.

| $f(t)$ | $y_p(t)$ |
|---|---|
| $e^{rt}$ | $ae^{rt}$ |
| $\cos \omega t$ or $\sin \omega t$ | $a \cos \omega t + b \sin \omega t$ |
| $n^{th}$ degree polynomial | $n^{th}$ degree polynomial |
| $e^{rt}\cos \omega t$ or $e^{rt}\sin \omega t$ | $e^{rt}(a \cos \omega t + b \sin \omega t)$ |
| linear combin. of above f$^{ns}$ | linear combin. of above f$^{ns}$. |

- **Simple harmonic motion:** $x'' = -\omega^2 x$ has solution
  $x(t) = a \cos \omega t + b \sin \omega t$, which we can write as a single cosine wave, $x(t) = A \cos\left(\omega\left(t - \frac{\phi}{\omega}\right)\right)$,
  where $A = \sqrt{a^2 + b^2}$

  $\phi = \arctan(b/a)$

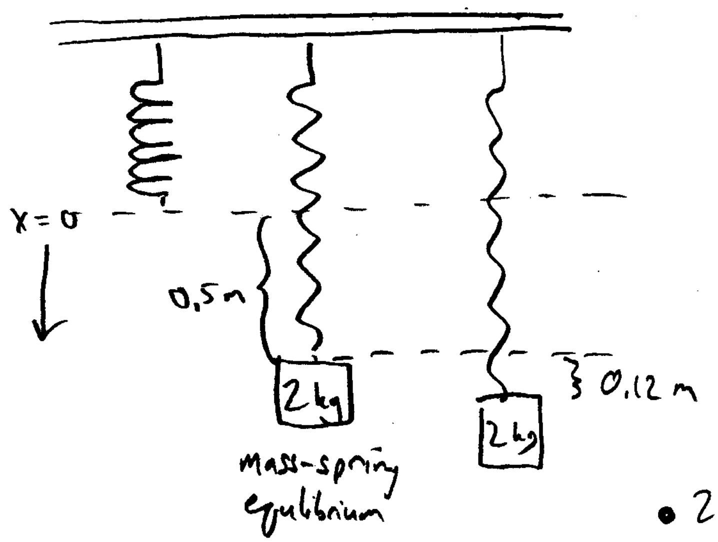



- **Harmonic motion** (in general, for mechanical systems).

$$\underbrace{mx''}_{\text{Newton's } 2^{nd}} + \underbrace{2cx'}_{\text{damping force}} + \underbrace{\omega_0^2 x}_{\text{spring force}} = \underbrace{f(t)}_{\text{driving force}}$$

Example: Simple harmonic motion + external (grav.) force.

A 2 kg mass is suspended from a spring. The displacement of the spring once the mass is attached is 50 cm. If the mass is displaced 12 cm downward from equilibrium, set up and solve the initial value problem that solves this.




• 1st, determine the spring constant: $kx_0 = mg$ (at equilibrium, spring force = grav. force).

$$\Rightarrow k = \frac{mg}{x_0} = \frac{2 \cdot 9.8}{0.5} = 39.2 \text{ N/m}$$

• 2nd: $F = mx'' = \sum$ forces.

$$\underbrace{mx''}_{\text{total force acting on the weight}} = \underbrace{-\mu x'}_{\substack{\text{damping}\\(=0)}} \underbrace{- kx}_{\text{spring}} + \underbrace{mg}_{\text{grav.}}$$

$$mx'' = -kx + mg$$

$$2x'' = -kx + kx_0 = -k(x - x_0) = -39.2(x - 0.5)$$

$$\boxed{2x'' + 39.2(x - 0.5) = 0, \quad x(0) = 0.62, \quad x'(0) = 0}$$

Let's solve this: $2x'' + 39.2x = 19.6$.

$x_h(t) = A\cos\omega t + B\sin\omega t$ where $\omega = \sqrt{19.6}$

$x_p(t) = 0.5$ (set $x'' = 0$ & solve for $x$).

General solution: $x(t) = A\cos\omega t + B\sin\omega t + 0.5$

Plug in $x(0) = 0.62$ & $x'(0) = 0$. ← this one look "easier," do it 1st.

$$x'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$x'(0) = 0 + B\omega = 0 \Rightarrow B = 0$$

$$x(t) = A\cos\omega t$$

$$x(0) = A = 0.62$$

$$\Rightarrow \boxed{x(t) = 0.12\cos(\sqrt{19.6}\,t) + 0.5}$$

<u>Next case: $c \neq 0$</u> (<u>damped</u> harmonic motion)

$$x'' + 2cx' + \omega_0^2 x = 0 \qquad c > 0$$

Assume $x(t) = e^{rt} \Rightarrow r^2 + 2cr + \omega_0^2 = 0$

$$\Rightarrow r = -c \pm \sqrt{c^2 - \omega_0^2}$$

<u>3 cases</u>:

(i) Complex roots ($c < \omega_0$): <u>under damped</u>:

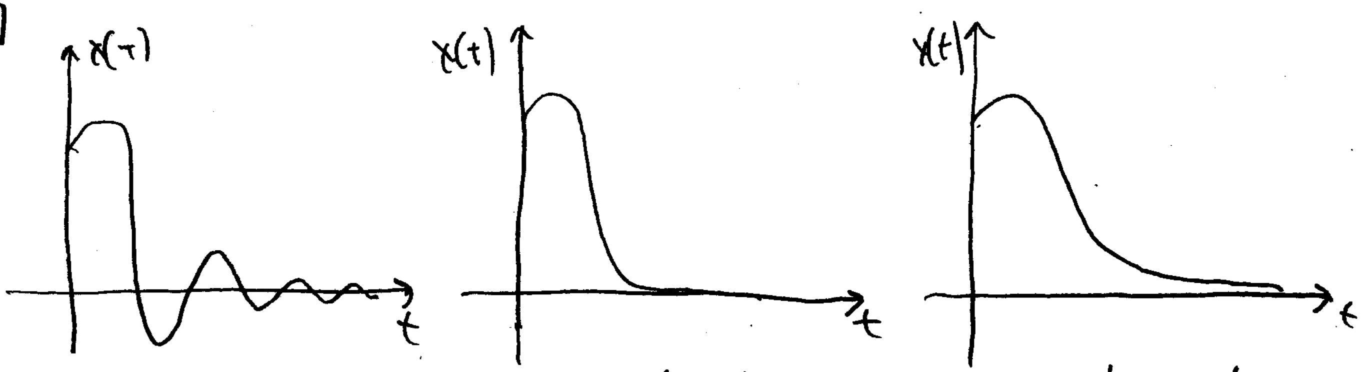
$$x(t) = e^{-ct}(a\cos\omega_0 t + b\sin\omega_0 t)$$

(ii) Double root ($c = 0$) <u>critically damped</u>

$$x(t) = C_1 e^{rt} + C_2 t e^{rt} \qquad (\text{note: } r_1 = r_2 < 0)$$

(iii) 2 real roots ($c > \omega_0$) <u>overdamped</u>.

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$$




underdamped | critically damped | overdamped.

**Next case:** Forced harmonic motion

e.g., • spring attached to a motor
• Source voltage is sinusoidal.

$$x'' + 2cx' + \omega_0 x = A\cos \omega t$$

damping coeff. | natural frequency | driving frequency

**Sample case:** No damping $(c=0)$

$$x'' + \omega_0^2 x = A\cos \omega t$$

Homog. eq'n: $x_h'' + \omega_0^2 x = 0$

$$x_h(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

**Case 1:** $\omega \neq \omega_0$.

$x_p = a\cos\omega t + b\sin\omega t$. Need to solve for $a, b$.

plug $x_p$ back in:

$$x_p'' + \omega_0^2 x_p = \overbrace{a(\omega_0^2-\omega^2)}^{=A}\cos\omega t + \overbrace{b(\omega_0^2-\omega^2)}^{=0}\sin\omega t$$

$$= A\cos\omega t + 0\sin\omega t$$

$$\Rightarrow \left.\begin{aligned} a(\omega_0^2-\omega^2) &= A \\ b(\omega_0^2-\omega^2) &= 0 \end{aligned}\right\} \Rightarrow a = \frac{A}{\omega_0^2-\omega^2}, \quad b=0.$$

i.e, $X_p(t) = \frac{A}{\omega_0^2-\omega^2}\underline{\cos}\ \omega t$ (Note: $A \to \infty$ as $\omega_0 \to \omega$ !)

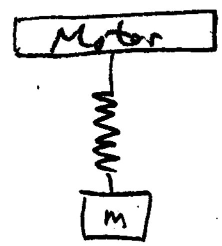
<u>General solution</u>: $X(t) = X_h(t) + X_p(t)$

$$= \boxed{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\omega_0^2-\omega^2}\cos \omega t}$$

What does this solution look like?

First, consider equilibrium: $\left.\begin{aligned} X(0)&=0 \\ X'(0)&=0 \end{aligned}\right\} \longrightarrow$ (See picture)
(for simplicity)

apply $F(t)$




equilibrium

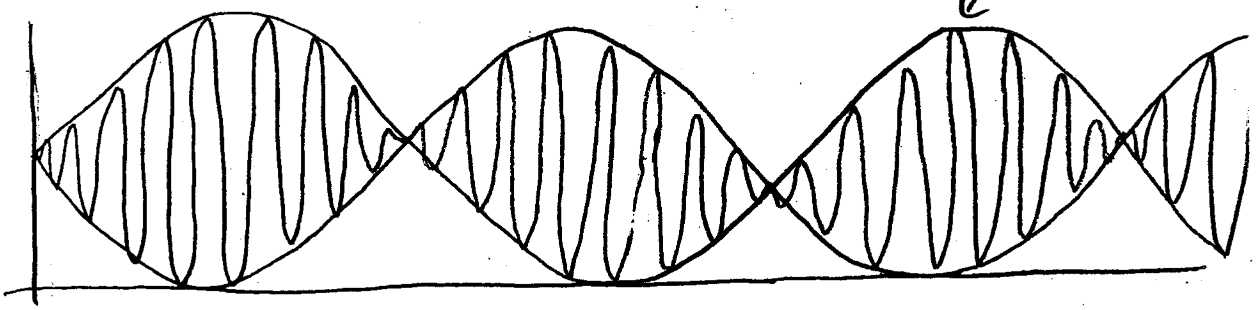
$$C_2 = 0, \quad C_1 = \frac{-A}{\omega_0-\omega^2} \Rightarrow X(t) = \frac{A}{\omega_0^2-\omega^2}(\cos \omega t - \cos \omega_0 t)$$

- Superposition of waves with different frequencies.

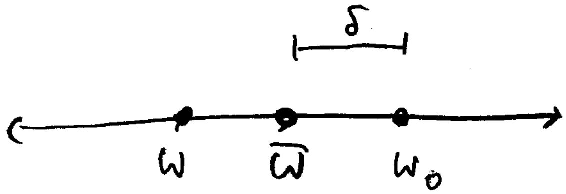
Has anyone experienced this in real life? (Think music!)

<u>Example</u>: $X(t) = \cos(11t) - \cos(12t)$

envelope



How to quantify this?




Let $\bar{\omega} = \frac{\omega + \omega_0}{2}$ (average)

and say $\omega = \bar{\omega} - \delta$

$\omega_0 = \bar{\omega} + \delta$

$$\Rightarrow \quad X(t) = \frac{A}{\omega_0^2 - \omega^2}\left(\cos \omega t - \cos \omega_0 t\right) = \underbrace{\left(\frac{A \sin \delta t}{2\bar{\omega}\, \delta}\right)}_{\text{Amplitude is sinusoidal.}}\left(\cos \omega t - \cos \omega_0 t\right).$$

<u>Case 2:</u> $\omega = \omega_0$ (Recall: $f(t) = \cos \omega t$ is the forcing term).

The ODE is now $x'' + \omega_0 x'' = A \cos \omega_0 t$

But $x_p(t) = A \cos \omega_0 t$ solves the homog. eq'n (the "problem case").

So we must try $x_p(t) = t\left(a \cos \omega_0 t + b \sin \omega_0 t\right)$

Plug $x_p$ back in:

$$x_p'' + \omega_0^2 x_p = \left[2\omega_0\left(-a \sin \omega_0 t + b \cos \omega_0 t\right) + t\omega_0^2\left(-a \cos \omega_0 t + b \sin \omega_0 t\right)\right]$$
$$+ \omega_0 t\left(a \cos \omega_0 t + b \sin \omega_0 t\right)$$
$$= 2\omega_0\left(-a \sin \omega_0 t + b \cos \omega_0 t\right) = \underbrace{A \cos \omega t + 0 \sin \omega t}_{\text{forcing term}}$$

$$\left.\begin{aligned} -2\omega_0 a &= 0 \\ 2\omega_0 b &= A \end{aligned}\right\} \Rightarrow \quad a = 0, \quad b = \frac{A}{2\omega_0}$$

Thus $x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t$

<u>General solution:</u> $x(t) = x_h(t) + x_p(t)$

$$= \boxed{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \overbrace{\left(\frac{A}{2\omega_0} t\right)}^{\text{Amplitude} \to \infty\,!} \sin \omega_0 t}$$

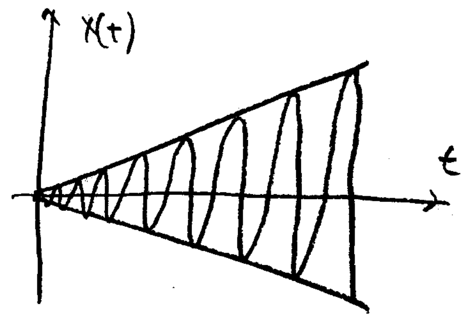
Look at the long-term behavior. This wave "blows up"!

Example: Again, consider starting at equilibrium: $x(0)=0$, $x'=0$.

$$x(0) = C_1 = 0. \qquad x'(t) = C_2 \omega_0 \cos \omega_0 t + \frac{A}{2} t \cos \omega_0 t + \frac{A}{2\omega_0} \sin \omega_0 t$$

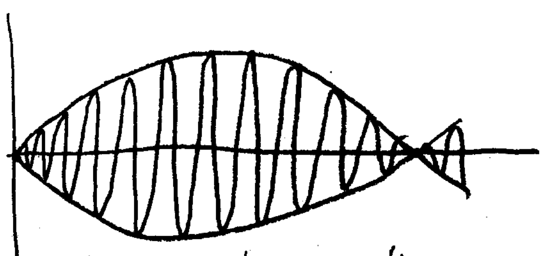
$$x'(0) = C_2 = 0 \implies \boxed{x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t}$$

Real-life example: Tacoma Narrows Bridge




envelope never "closes up"
(like dividing by zero).

Compare this to $\omega_0 \approx \omega$



envelope "closes up"

## Next topic: Systems of ODE's:

Recall how we can turn a $2^{nd}$ order ODE into a system of two $1^{st}$ order ODE's:

e.g., $x'' + 3x' - x = \sin 3t$. let $y = x' \implies y' = x''$.

Plug back in: $\begin{cases} y' + 3x' - x = \sin 3t \\ y = x' \end{cases}$

One benefit of doing this: Many numerical methods (like Euler's method) don't work for higher order ODE's, but <u>do</u> work for systems of $1^{st}$ order ODE's.

First, we need to learn some basic linear algebra ("matrix algebra").