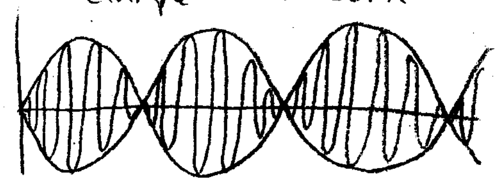
Week 6 summary:

- Harmonic motion: $x'' + 2cx' + \omega_0^2 x = f(t)$
  - Simple harmonic motion: $(c = 0,\ f(t) = 0)$.
    $x(t) = a\cos\omega_0 t + b\sin\omega_0 t$ or $A\cos(\omega_0 t - \phi)$
  - With damping $(c \neq 0)$. Roots $r_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2}$
    - case 1: $c > \omega_0$ overdamped $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
    - case 2: $c = \omega_0$ critically damped $x(t) = C_1 e^{rt} + C_2 t e^{rt}$
    - case 3: $c < \omega_0$ underdamped $x(t) = e^{-ct}(a\cos\omega_0 t + b\sin\omega_0 t)$
  - Forced harmonic motion $(f(t) \neq 0)$.

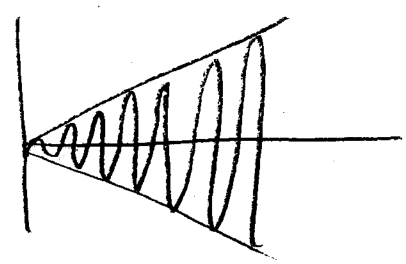
    e.g., $x'' + \omega_0^2 x = A\cos\omega t$

    example IVP sol'n

    - case 1: $\omega \neq \omega_0$ $x(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t) = \frac{A}{2\delta\bar{\omega}}\sin\delta t\sin\bar{\omega}t$
    - case 2: $\omega = \omega_0$ $x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{A}{2\omega_0} t\sin\omega_0 t$

    example IVP sol'n:

- We can write a 2^{nd} order ODE into a system of two 1^{st} order ODEs.

This week: Basic linear algebra ("matrix algebra").

Basic Matrix algebra

Matrices add in the "obvious way:" $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$

Multiplication is slightly more complicated:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

We can <u>only</u> multiply an $n \times m$ matrix ("n rows, m columns") by an $m \times r$ matrix. The result is an $n \times r$ matrix.

<u>Examples:</u>

$$\underset{2\times 2 \quad 2\times 1}{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}}, \qquad \underset{2\times 3 \quad 3\times 1}{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} ax+by+cz \\ dx+ey+fz \end{pmatrix}}$$

<u>What doesn't work:</u>

$$\underset{2\times 2 \quad 3\times 2}{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}, \qquad \underset{3\times 1 \quad 2\times 2}{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} d & e \\ f & g \end{pmatrix}}$$

<u>Def:</u> An $n \times 1$ matrix is called a <u>vector</u>.

<u>Application of matrices:</u> Systems of linear algebraic equations.

Consider the system $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$ (x_1 & x_2 are the "unknowns")

We can write this as:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \text{or} \quad A\vec{x} + \vec{b}$$

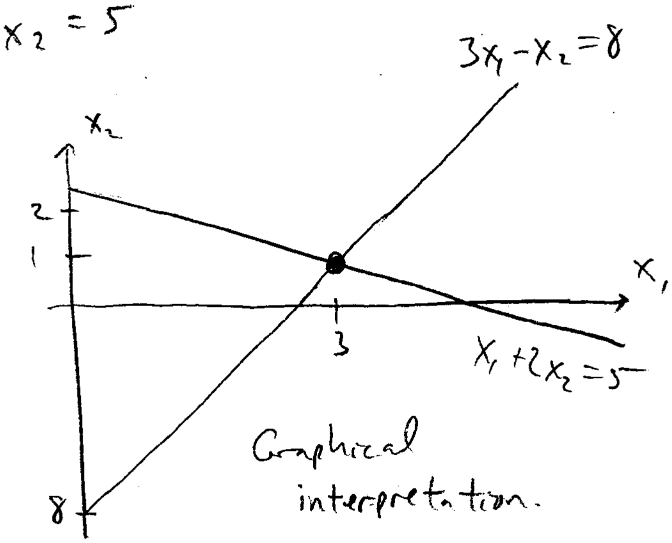
<u>Goal:</u> Solve for $\vec{x}$.

Example 1: Solve the system $\begin{cases} 3x_1 - x_2 = 8 \\ x_1 + 2x_2 = 5 \end{cases}$

This is easy: $x_2 = 3x_1 - 8$

Sub. back in, get $x_1 + 2(3x_1 - 8) = 5$

$x_1 = 3$, $x_2 = 1$.



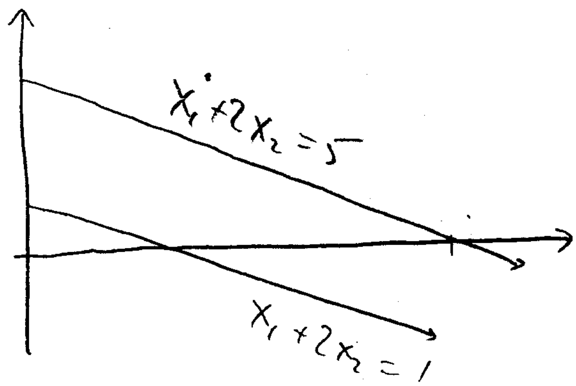
Graphical interpretation.

Example 2: Solve the system $\begin{cases} x_1 + 2x_2 = 1 \\ x_1 + 2x_2 = 5 \end{cases}$

$x_1 = 1 - 2x_2$.

Sub. back in, get $1 = 5$, which is false, thus there is no solution.

(i.e., no values of x_1 & x_2 satisfy these equations.

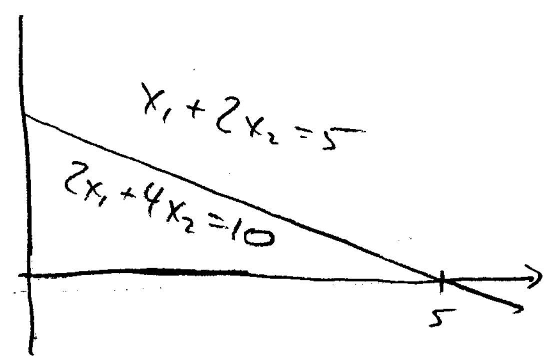


Example 3: Solve the system $\begin{cases} 2x_1 + 4x_2 = 10 \\ x_1 + 2x_2 = 5 \end{cases}$

$x_1 = 5 - 2x_2$

Sub. back in, get $2(5 - 2x_2) + 4x_2 = 10$, or $10 = 10$.

(i.e, any values of x_1 & x_2 satisfy these equations).

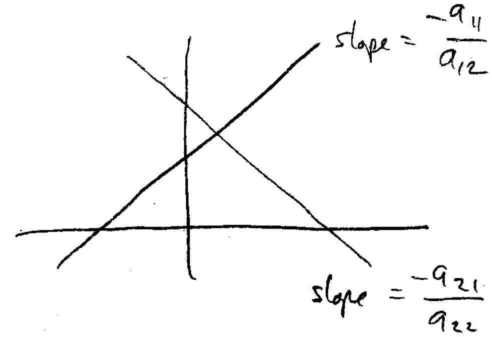


Consider the general case: $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

Solving for x_1, we get $x_1 = \dfrac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$, $x_2 = \dfrac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$

Note: There will be a unique solution for x_1 & x_2 if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

This quantity is called the determinant of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and is denoted $\det A$, or $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.



Geometric intrepretation:

If the slopes are different, then there will be a unique solution, and $\frac{-a_{11}}{a_{12}} \neq \frac{-a_{21}}{a_{22}}$, i.e., $a_{11}a_{22} - a_{12}a_{21} = \det A \neq 0$.

Def: The (2×2) identity matrix is $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

This is the "matrix analog" of the multiplicative identity, 1.

Next, we'll look for a matrix analog of "inverses", i.e., the fact that for any fraction $\frac{a}{b}$: $\frac{a}{b} \cdot \frac{b}{a} = 1$, (the multiplicative identity).

* Given A, we want to find a matrix B such that $AB = BA = I$.
Such a matrix is called the inverse of A, and denoted A^{-1}.

Note that $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$

$= (\det A) \cdot I.$

Thus, $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$

When A^{-1} exists, we say that A is <u>invertible</u>, or <u>nonsingular</u>.
Otherwise, A is <u>noninvertible</u>, or <u>singular</u>.

* Thus, A is invertible if and only if $\det A \neq 0$.

<u>Back to systems:</u> We want to solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

If A is invertible, then write $A^{-1}A\vec{x} = A^{-1}\vec{b}$

$\Rightarrow \boxed{\vec{x} = A^{-1}\vec{b}}$

<u>Recall example 1:</u> $\begin{cases} 3x_1 - x_2 = 8 \\ x_1 + 2x_2 = 5 \end{cases}$

Write as $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}.$

$A \quad \vec{x} \quad = \vec{b}$

$\det A = 3 \cdot 2 - (-1) \cdot 1 = 7$, so $A^{-1} = \frac{1}{7} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}.$

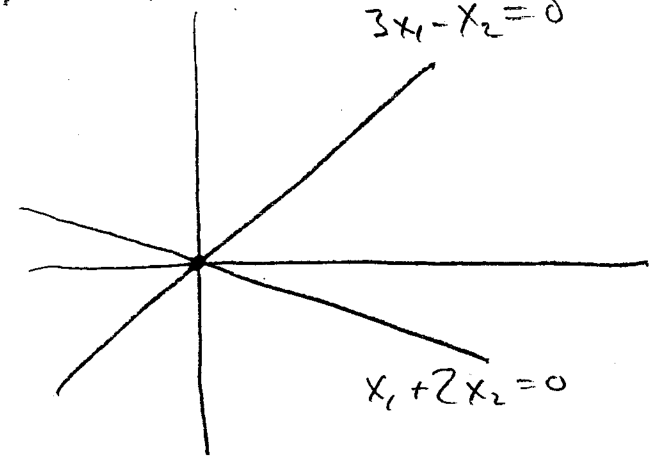
The solution is thus $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 21 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\vec{x} \quad = \quad A^{-1} \cdot \vec{b}$

i.e, $x_1 = 3$, $x_2 = 1$ ✓

Def: If $b_1 = b_2 = 0$ in a system, then the system is <u>homogeneous</u>.
In matrix notation, this is $A\vec{x} = \vec{0}$.

Graphical intrepretation: 2 lines through the origin, $\vec{0}$.



Note: If $\det A \neq 0$, then the <u>only</u> solution is $x_1 = x_2 = 0$,
i.e, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Geometrically, this is clear.
Algebraically, $A\vec{x} = \vec{0} \implies A^{-1}A\vec{x} = A\vec{0} \implies \vec{x} = 0$.

<u>A geometric way to view matrices:</u>
A 2×2 matrix is a linear map from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

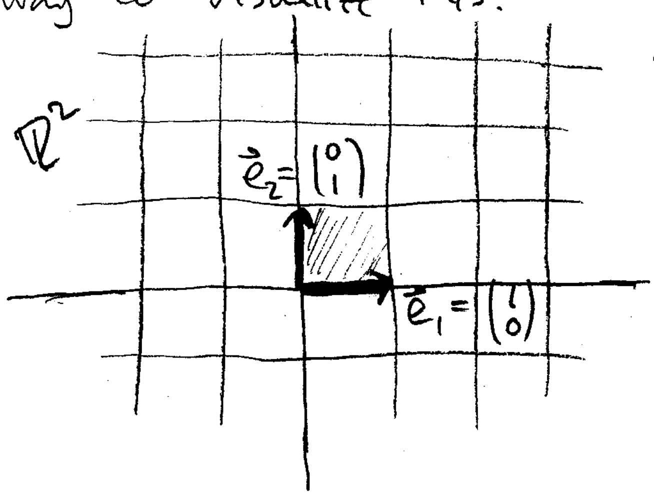
$$A : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \text{i.e,} \quad A\vec{x} = \vec{y}$$

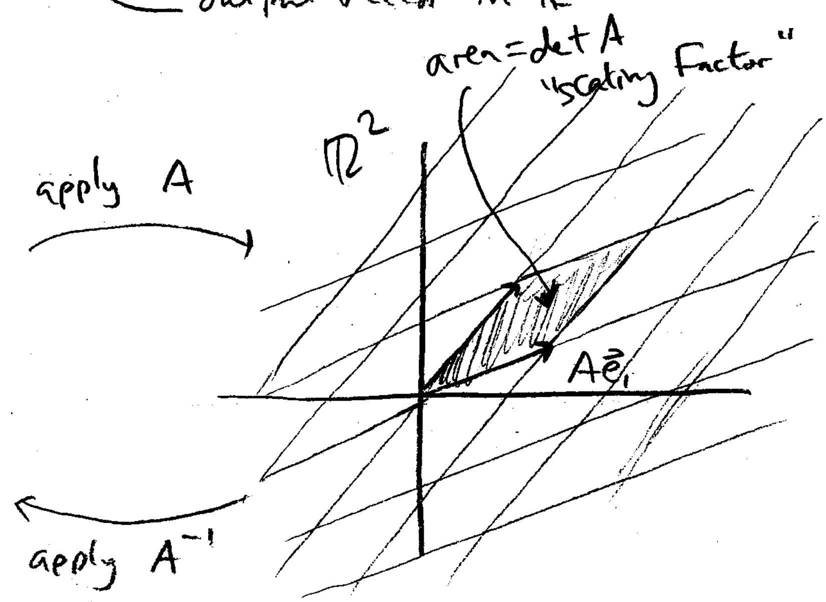
explicitly, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{21}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$

input vector in $\mathbb{R}^2$

output vector in $\mathbb{R}^2$

way to visualize this.





Imagine "picking up $\vec{e}_1$ & $\vec{e}_2$ and moving them to $A\vec{e}_1$ & $A\vec{e}_2$, and the grid "comes along with them"!

Note: $A\vec{e}_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, $\quad A\vec{e}_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$.

Visually, A^{-1} is the map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that "undos" A.

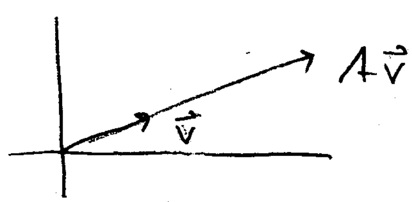
Fact: $\det A$ = area of the image of the unit square under A.

Note: If this area $=0$, then the grid has "collapsed" to 1-D, and we can't "undo" (or "pull apart") this. (This is a geometric intrepretation of why if $\det A = 0$, then A^{-1} doesn't exist)

Eigenvalues and eigenvectors:

Def: Let A be a matrix. A vector $\vec{v}$ is an eigenvector of A if $A\vec{v} = \lambda\vec{v}$ for some constant λ, which is called an eigenvalue.

Geometrically:



How to find them:

We want to solve $A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} - \lambda\vec{v} = 0$
$\Rightarrow (A - \lambda I)\vec{v} = 0$

This equation is homogeneous, so the only solution is $\vec{v} = 0$, unless $\det(A - \lambda I) = 0$.

So we must solve $\det(A-\lambda I)=0$ for λ.

$$A-\lambda I = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = \underbrace{(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}}_{\text{polynomial in } \lambda} = 0$$

$\Rightarrow \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22}-a_{12}a_{21}) = 0$ "characteristic polynomial"

Def: The trace of A is the sum of the diagonal entries, and denoted $\operatorname{tr} A$.

If A is 2×2, then $\operatorname{tr} A = a_{11}+a_{22}$.

Remark: The characteristic polynomial is $\boxed{\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = 0}$

The roots of the char. poly. are the eigenvalues of A.

Revisit example 1: (Real, distinct eigenvalues)

Find the eigenvalues of $A=\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0.$$

$\Rightarrow \boxed{\lambda_1 = 3 \text{ and } \lambda_2 = -1}$

Now, let's find the associated eigenvectors, $\vec{v}_1$ and $\vec{v}_2$.

$\boxed{\lambda_1=3}$: The eigenvector $\vec{v}_1$ solves $(A-3I)\vec{v} = \vec{0}$.

$$(A-3I)\vec{v} = \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e, $\begin{cases} -2x_1 + x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{cases}$ This system is <u>redundant</u> (because $\det(A-3I)=0$)

Thus, $-2x_1 + x_2 = 0 \Rightarrow x_2 = 2x_1$

"x_2 is twice as big as x_1"

So, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ 2c \end{pmatrix} = c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

* Any of these is an eigenvector, so let's just pick one: $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Thus, $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A, with eigenvalue $\lambda_1 = 3$.

i.e, $A\vec{v}_1 = 3\vec{v}_1$.

<u>Check</u>: $A\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\vec{v}_1$ ✓

Let's find the other eigenvector

$\boxed{\lambda_2 = -1}$: The eigenvector $\vec{v}_2$ solves $(A+I)\vec{v} = \vec{0}$.

$$(A+I)\vec{v} = \begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ 4x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

i.e, $\begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{cases}$ Again, the equations are redundant, so we just need to consider one of them.

$2x_1 + x_2 = 0 \Rightarrow x_2 = -2x_1$

$\Rightarrow \vec{v} = \begin{pmatrix} c \\ -2c \end{pmatrix} = c\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector for any c.

Let's just pick one, say $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

<u>Check</u>: $A\vec{v}_2 = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -\vec{v}_2$ ✓

* In summary, the matrix A has eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ & $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and associated eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$.

Example 2: (Complex eigenvalues)

Find the eigenvalues of $A = \begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{2} - \lambda & 1 \\ -1 & -\frac{1}{2} - \lambda \end{vmatrix} = \left(-\tfrac{1}{2} - \lambda\right)^2 + 1 = \lambda^2 + \lambda + \tfrac{5}{4} = 0$$

$$\Rightarrow \boxed{\lambda_{1,2} = -\tfrac{1}{2} \pm i}$$

Find the eigenvectors:

$\boxed{\lambda_1 = -\frac{1}{2} + i}$: The eigenvector solves $(A - \lambda I)\vec{v} = \vec{0}$

$$\left(A - \left(-\tfrac{1}{2} + i\right) I\right)\vec{v} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -i x_1 + x_2 = 0 \\ -x_1 - i x_2 = 0 \end{cases}$$

Again, the eq'ns are redundant, and so $-i x_1 + x_2 = 0$

$\Rightarrow x_2 = i x_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} c \\ ic \end{pmatrix} = c\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector.

$\boxed{\lambda_2 = -\frac{1}{2} - i}$:

$$\left(A - \left(-\tfrac{1}{2} - i\right) I\right)\vec{v} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} i x_1 + x_2 = 0 \\ -x_1 + i x_2 = 0 \end{cases}$$

Redundant eq'ns $\Rightarrow i x_1 + x_2 = 0 \Rightarrow x_2 = -i x_1$

$\Rightarrow v_2 = \begin{pmatrix} c \\ -ic \end{pmatrix} = c\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector.

* In summary, the matrix A has eigenvalues $\lambda_1 = -\frac{1}{2} + i$, $\lambda_2 = -\frac{1}{2} - i$, and associated eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.