MTHSC 208, WEEK 8

Week 7 summary

Basic linear algebra

- A system of 2 linear equations $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

  can be written as $A\vec{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \vec{b}$.

- $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ iff $\underline{\det A} := a_{11}a_{22} - a_{12}a_{21} \neq 0$.

  The homogeneous eq'n $A\vec{x} = \vec{0}$ has a nonzero sol'n iff $\det A = \vec{0}$.

- The inverse of A exists iff $\det A \neq 0$, and is $A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{pmatrix}$,

  and $AA^{-1} = A^{-1}A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the "identity matrix."

- If $A\vec{v} = \lambda\vec{v}$, then $\vec{v}$ is an <u>eigenvector</u> for A with eigenvalue λ.

  To find λ, solve $\det(A - \lambda I) = 0$.

  <u>Note:</u> $\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = 0$, $\quad \operatorname{tr} A = a_{11} + a_{22}$.

  Then, to find $\vec{v}$, solve $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v}$.

Finding eigenvectors & eigenvalues (cont.)

<u>Example 3</u>: (Repeated eigenvalues, <u>one</u> eigenvector). Consider $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

$\boxed{\lambda = 2}$ (multiplicity 2). Find eigenvector.

Solve $(A - 2I)\vec{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$

Thus $\begin{pmatrix} -c \\ c \end{pmatrix}$ is an eigenvector, so "pick one," say $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We have $\boxed{\lambda = 2, \quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

Example 4: (Repeated eigenvalue, multiple eigenvectors). Consider $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 \Rightarrow \lambda = 2 \text{ (multiplicity 2)}.$$

$\boxed{\lambda = 2}$ Find eigenvectors.

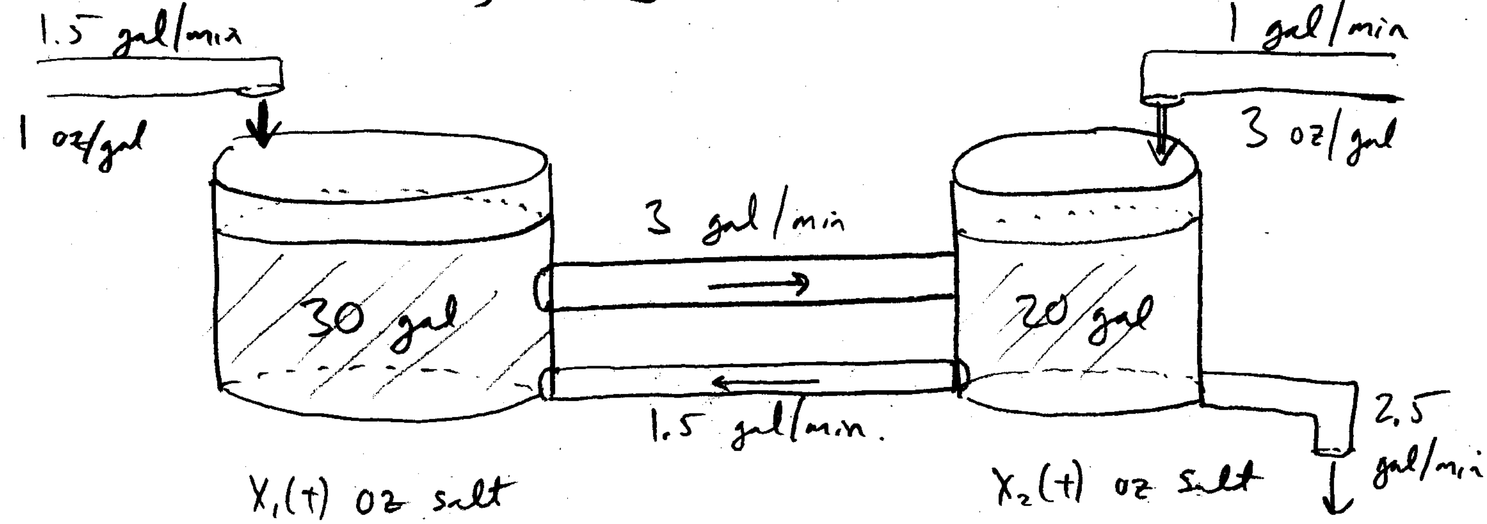
$$(A - 2I)\vec{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 0x_1 + 0x_2 = 0 \\ 0x_1 + 0x_2 = 0 \end{cases}$$

x_1 & x_2 can be anything!

Thus, every vector is an eigenvector. (This is quite "unusual")

Systems of 2 linear 1st order ODEs:

Consider the following mixing problem:



$x_1(t)$ oz salt $\qquad$ $x_2(t)$ oz salt

Initially: $x_1(0) = 55$ $\qquad$ $x_2(0) = 26$

Note: Volume of both tanks stays constant.

Tank 1: rate in $= \left(1.5 \frac{gal}{min}\right)\left(\frac{x_2(t) \text{ oz}}{20 \text{ gal}}\right) + \left(1.5 \frac{gal}{min}\right)\left(1 \frac{oz}{min}\right) = 0.075 x_2 + 1.5$

rate out $= \left(3 \frac{gal}{min}\right)\left(\frac{x_1(t) \text{ oz}}{30 \text{ gal}}\right) = 0.1 x_1$

Tank 2: rate in $= \left(3 \frac{gal}{min}\right)\left(\frac{x_1(t) \text{ oz}}{30 \text{ gal}}\right) + \left(1 \frac{gal}{min}\right)\left(3 \frac{oz}{gal}\right) = 0.1 x_1 + 3$

rate out $= \left(4 \frac{gal}{min}\right)\left(\frac{x_2(t) \text{ oz}}{20 \text{ gal}}\right) = 0.2 x_2$

Tank 1: $x_1' = (\text{rate in}) - (\text{rate out}) = -0.1x_1 + 0.075x_2 + 1.5$

$x_2' = (\text{rate in}) - (\text{rate out}) = 0.1x_1 - 0.2x_2 + 3$

we now have a system: $\vec{x}' = K\vec{x} + \vec{b}$, $\vec{x}(0) = \begin{pmatrix} 55 \\ 26 \end{pmatrix}$.

$$\begin{cases} \dfrac{dx_1}{dt} = -0.1x_1 + 0.075x_2 + 1.5 \\[2ex] \dfrac{dx_2}{dt} = 0.1x_1 - 0.2x_2 + 3 \end{cases} \Longrightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

with initial condition $\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 55 \\ 26 \end{pmatrix}$.

The variables x_1 & x_2 are the <u>state variables</u>.

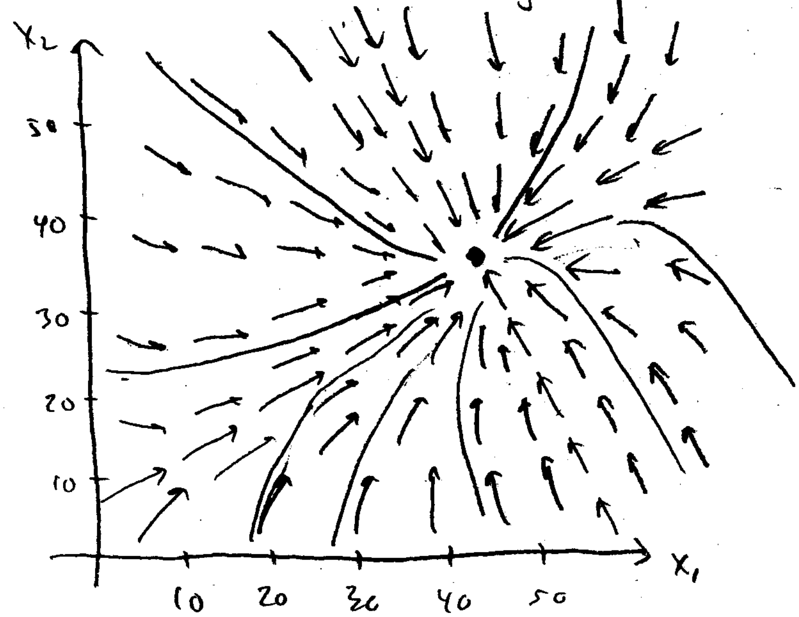
The vector $\vec{x} = x_1\vec{i} + x_2\vec{j} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the <u>state vector</u> of the system.

The x_1x_2-plane is the <u>state plane</u>, or <u>phase plane</u>.

<u>Note</u>: $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a curve in the phase plane.

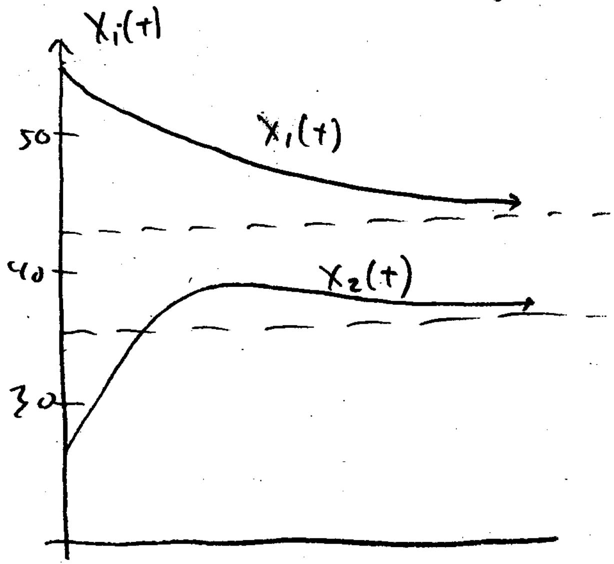
(Think: what does this curve describe?)

There are several ways to visualize solutions to this system:



Phase portrait for the system

Note the "steady-state" sol'n at (42, 36)



Component plots for the IVP.

4

This system is <u>autonomous</u>, i.e., neither K nor $\vec{b}$ (in $\vec{x}' = K\vec{x} + \vec{b}$) depend on t.

Thus, as before, we can set $\vec{x}' = \vec{0}$ and solve for $\vec{x}$ to find a constant <u>steady-state</u> solution:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ We must solve $\begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1.5 \\ -3 \end{pmatrix}$

$$\det K = \frac{1}{80} \Rightarrow K^{-1} = 80\begin{pmatrix} -0.2 & -0.075 \\ -0.1 & -0.1 \end{pmatrix} = \begin{pmatrix} -16 & -6 \\ -8 & -8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = K^{-1}\begin{pmatrix} -1.5 \\ -3 \end{pmatrix} = \begin{pmatrix} -16 & -6 \\ -8 & -8 \end{pmatrix}\begin{pmatrix} -1.5 \\ -3 \end{pmatrix} = \begin{pmatrix} 42 \\ 36 \end{pmatrix}$$

Thus $\boxed{x_1 = 42,\ x_2 = 36}$ is the steady-state sol'n.

Now, let's change variables: let $y_1(t) = x_1(t) - 42$, $y_2(t) = x_2(t) - 36$.

$\Rightarrow x_1 = y_1 + 42$, $x_2 = y_2 + 36$ and $x_1' = y_1'$, $x_2' = y_2'$.

Plug back into our system: $\begin{cases} x_1' = -0.1\, x_1 + 0.075\, x_2 + 1.5 \\ x_2' = 0.1\, x_1 + 0.2\, x_2 + 3 \end{cases}$

and we get a <u>homogeneous</u> system: $\begin{cases} y_1' = -0.1\, y_1 + 0.075\, y_2 \\ y_2' = 0.1\, y_1 + 0.2\, y_2 \end{cases}$

i.e, $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & 0.2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

$$\vec{y}' = K\,\vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$$

Now, we just have to solve the <u>homogeneous</u> eq'n $\vec{y}' = K\vec{y}$,
and then add $\mathring{\vec{x}}_p(t) = \begin{pmatrix} 42 \\ 36 \end{pmatrix}$ back to it.

* Yes, we're really just doing $\boxed{\vec{x} = \vec{x}_h + \vec{x}_p}$ here!

Graphically, the change of variables "brings the steady-state" point $\begin{pmatrix} 42 \\ 36 \end{pmatrix}$ in the phase portrait back to the origin, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

<u>Solving a homogeneous system of ODE's</u>

Consider a simple example: $\vec{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \vec{x}$, $\quad \vec{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

This is the system $\begin{cases} x_1' = -x_1 & x_1(0) = 2 \\ x_2' = -4x_2 & x_2(0) = 3 \end{cases}$

This is easy! The general solution is $x_1(t) = C_1 e^{-t}$, $x_2(t) = C_2 e^{-4t}$.

Writing this in "vector notation", and we get

$$\vec{x} = \begin{pmatrix} C_1 e^{-t} \\ C_2 e^{-4t} \end{pmatrix} = C_1 \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ e^{-4t} \end{pmatrix} = \boxed{C_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Plug in initial conditions: $\vec{x}(0) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\Rightarrow C_1 = 2,\ C_2 = 3 \Rightarrow \boxed{\vec{x}(t) = 2e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

<u>Moral</u>: If A is diagonal, then solving this system is easy!

Let's consider the more general case: A is <u>any</u> 2×2 matrix.

Suppose $\vec{v}$ is an eigenvector of A with eigenvalue λ.

* <u>Claim:</u> $\vec{x}(t) = e^{\lambda t}\vec{v}$ solves $\vec{x}' = A\vec{x}$.

<u>Proof:</u> (Easy!) Just plug it in and check:

If $\vec{x}(t) = e^{\lambda t}\vec{x} = \begin{pmatrix} e^{\lambda t} x_1 \\ e^{\lambda t} x_2 \end{pmatrix}$, then $\vec{x}'(t) = \begin{pmatrix} \lambda e^{\lambda t} x_1 \\ \lambda e^{\lambda t} x_2 \end{pmatrix} = \lambda e^{\lambda t}\vec{x}$.

Thus, $\vec{x}' = \underline{\lambda e^{\lambda t}\vec{x}}$ and $A\vec{x} = e^{\lambda t}A\vec{v} = \underline{e^{\lambda t}\lambda\vec{x}}$

i.e, $\vec{x}' = A\vec{x}$. ✓

Since $\vec{x}' = A\vec{x}$ is homogeneous, then if $\vec{x}_1$ & $\vec{x}_2$ are solutions, so is $C_1\vec{x}_1 + C_2\vec{x}_2$. ("Superposition.")

<u>Conclusion:</u> Suppose A has distinct real eigenvalues $\lambda_1 \neq \lambda_2$ and eigenvectors $\vec{v}_1$, $\vec{v}_2$. Then the general solution to $\vec{x}' = A\vec{x}$ is $\boxed{\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}$

Not surprisingly, specifying an initial condition $\vec{x}(0) = \vec{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$ determines a unique solution.

The case is a little trickier if A has complex eigenvalues, or a repeated eigenvalue.

Return to solving $\vec{y}' = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & 0.2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{y}(0) = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

It is easily verified that the eigenvalues & eigenvectors are

$\lambda_1 = -0.25$, $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\lambda_2 = -0.05$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Thus, the general solution is $\boxed{\vec{y}(t) = c_1 e^{-0.25t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-0.05t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$

Plug in initial conditions: $\vec{y}(0) = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

i.e., $\begin{cases} c_1 + 3c_2 = 13 \\ -2c_1 + 2c_2 = -10 \end{cases}$ or just $\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

This has solution $c_1 = 7$, $c_2 = 2$

Thus, we get a unique solution $\boxed{\vec{y}(t) = e^{-0.25t}\begin{pmatrix} 7 \\ -14 \end{pmatrix} + e^{-0.05t}\begin{pmatrix} 6 \\ 4 \end{pmatrix}}$

i.e., $y_1(t) = 7e^{-0.25t} + 6e^{-0.05t}$

$y_2(t) = -14e^{-0.25t} + 4e^{-0.05t}$

$\Rightarrow$ $x_1(t) = 7e^{-0.25t} + 6e^{-0.05t} + 42$

$x_2(t) = -14e^{-0.25t} + 4e^{-0.05t} + 36$

<u>Next, let's plot this</u>

We'll plot $\vec{y}(t)$; it's centered at $\vec{0}$ so it's easier.

Then we just shift $\vec{y}(t)$ back to $(42, 36)$

(equivalently, add the particular sol'n $\vec{x}_p(t) = \begin{pmatrix} 42 \\ 36 \end{pmatrix}$).

Return to solving $\vec{y}\,' = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & 0.2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{y}(0) = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$.

It is easily verified that the eigenvalues & eigenvectors are

$$\lambda_1 = -0.25, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \lambda_2 = -0.05, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Thus, the general solution is $\boxed{\vec{y}(t) = C_1 e^{-0.25t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-0.05t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$

Plug in initial conditions: $\vec{y}(0) = C_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

i.e., $\begin{cases} C_1 + 3C_2 = 13 \\ -2C_1 + 2C_2 = -10 \end{cases}$ or just $\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

This has solution $C_1 = 7$, $C_2 = 2$

Thus, we get a unique solution $\boxed{\vec{y}(t) = e^{-0.25t}\begin{pmatrix} 7 \\ -14 \end{pmatrix} + e^{-0.05t}\begin{pmatrix} 6 \\ 4 \end{pmatrix}}$.

i.e.,
$$y_1(t) = 7e^{-0.25t} + 6e^{-0.05t}$$
$$y_2(t) = -14e^{-0.25t} + 4e^{-0.05t}$$

$$\Rightarrow \quad x_1(t) = 7e^{-0.25t} + 6e^{-0.05t} + 42$$
$$x_2(t) = -14e^{-0.25t} + 4e^{-0.05t} + 36$$

<u>Next, let's plot this</u>

We'll plot $\vec{y}(t)$; it's centered at $\vec{0}$ so it's easier.

Then we just shift $\vec{y}(t)$ back to $(42, 36)$

(equivalently, add the particular sol'n $\vec{x}_p(t) = \begin{pmatrix} 42 \\ 36 \end{pmatrix}$).

How do we plot $\vec{x}(t) = C_1 \vec{x}_1(t) + C_2 \vec{x}_2(t)$?

Let's try a few "special cases."

<u>Suppose $C_2 = 0$.</u> Then $\vec{x}(t) = C_1 \vec{x}_1(t) \begin{pmatrix} C_1 e^{-0.25t} \\ -2C_1 e^{-0.25t} \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

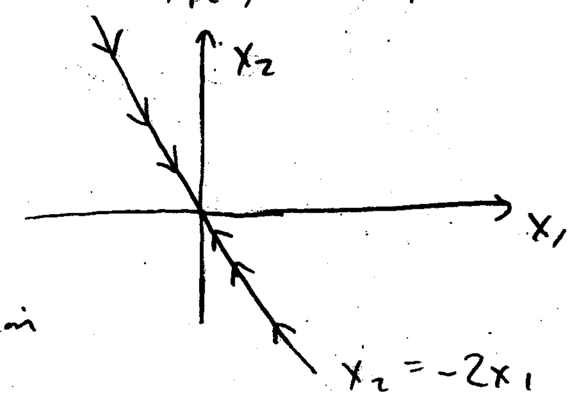
The x_1-value is $C_1 e^{-0.25t}$

The x_2-value is $-2C_1 e^{-0.25t}$

Thus, the <u>slope</u> of this solution is $\dfrac{x_2(t)}{x_1(t)} = \dfrac{-2C_1}{C_1} = -2$.

We've shown that $\dfrac{x_2}{x_1} = -2$

$\Rightarrow \boxed{x_2 = -2x_1}$



This means that there is a solution curve on the line $x_2 = -2x_1$

<u>Note</u>: $\lim\limits_{t\to\infty} x_1(t) = \lim\limits_{t\to\infty} C_1 e^{-0.25t} = 0$

$\lim\limits_{t\to\infty} x_2(t) = \lim\limits_{t\to\infty} -2C_1 e^{-0.25t} = 0$,

Therefore, as $t \to \infty$, the solution curves on this line move <u>toward</u> the origin,

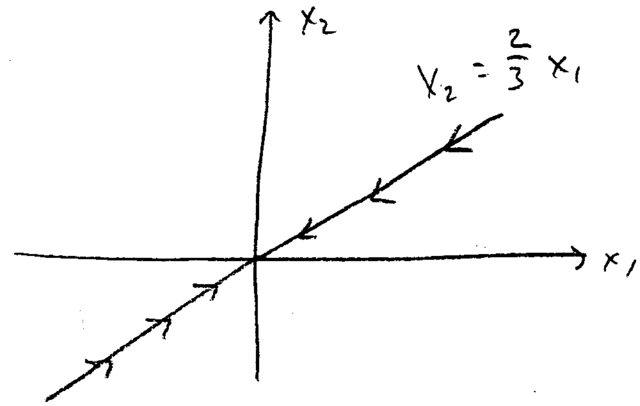
Next, <u>suppose $C_1 = 0$</u>. Then $\vec{x}(t) = C_2 \vec{x}_2(t) = \begin{pmatrix} 3C_2 e^{-0.05t} \\ 2C_2 e^{-0.05t} \end{pmatrix}$

The x_1-value is $3C_2 e^{-0.05t}$

The x_2-value is $2C_2 e^{-0.05t}$

Thus, the slope of this solution is $\frac{x_2(t)}{x_1(t)} = \frac{2c_2}{3c_2} = \frac{2}{3}$.

$\Rightarrow \frac{x_2}{x_1} = \frac{2}{3} \Rightarrow \boxed{x_2 = \frac{2}{3} x_1}$



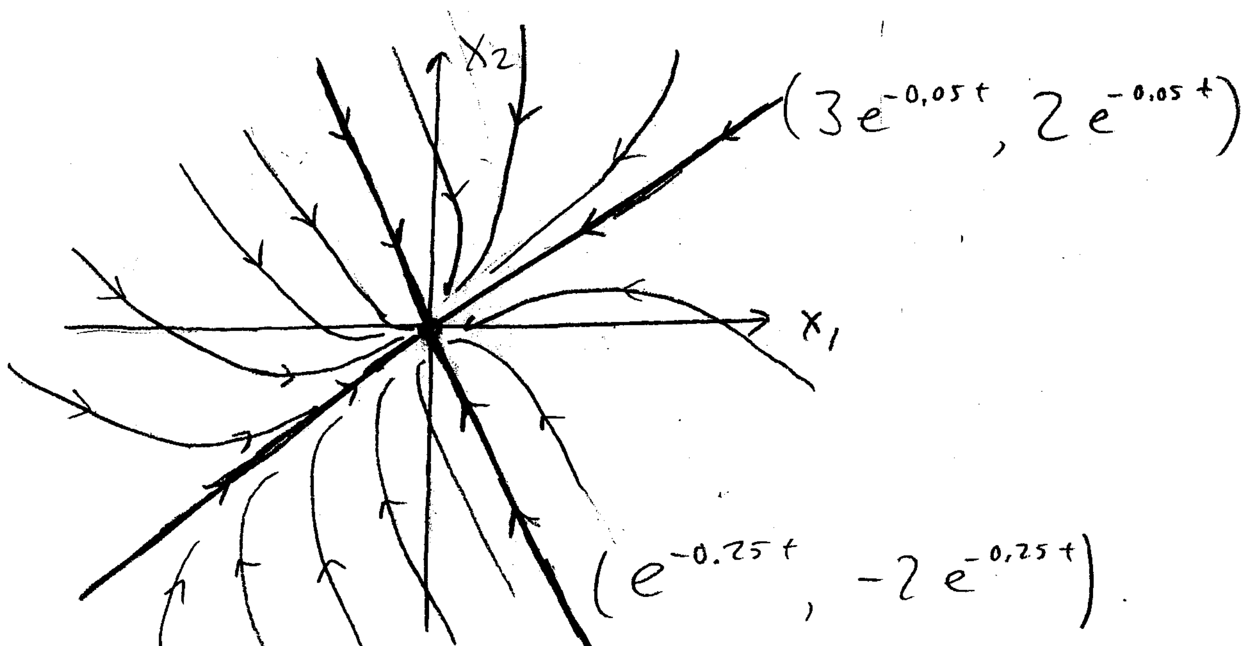
This means that there is a solution curve on $x_2 = \frac{2}{3} x_1$

Note: $\lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} 3c_2 e^{-0.05t} = 0$

$\lim_{t \to \infty} x_2(t) = \lim_{t \to \infty} 2c_2 e^{-0.05t} = 0$

Therefore, as $t \to \infty$, the solution curves on this line move <u>toward</u> the origin, but <u>slower</u> (Why?)

Together, we can now sketch the solutions to this system, via a <u>phase plot</u>:



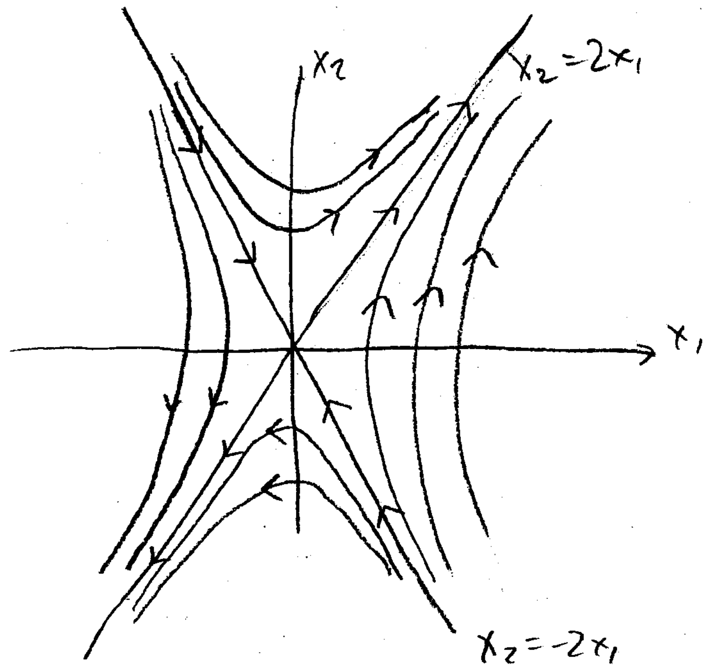
<u>Example</u> $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$ <u>Recall:</u> $\lambda_1 = 3$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -1$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Thus, the general soln is $\vec{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$,

i.e., $\begin{cases} x_1(t) = c_1 e^{3t} + c_2 e^{-t} \\ x_2(t) = 2c_1 e^{3t} - 2c_2 e^{-t} \end{cases}$

<u>Suppose $c_2 = 0$:</u> $x_1 = c_1 e^{3t}$, $x_2 = 2c_1 e^{3t}$ $\Rightarrow$ slope $= \frac{x_2}{x_1} = 2 \Rightarrow x_2 = 2x_1$



Along this line, $\lim_{t \to \infty} x_1(t) = \pm\infty$

Thus, the solutions move <u>away</u> from the origin

<u>Suppose $c_1 = 0$:</u> $x_1 = c_2 e^{-t}$, $x_2 = -2c_2 e^{-t}$

$\Rightarrow$ slope $= \frac{x_2}{x_1} = -2 \Rightarrow x_2 = -2x_1$

Along this line, $\lim_{t \to \infty} x_1(t) = 0$.

Thus, the solutions move <u>toward</u> the origin.

Summary: Suppose $\vec{x}' = A\vec{x}$ and A has real eigenvalues $\lambda_1 \neq \lambda_2$ with eigenvectors $\vec{v}_1$, $\vec{v}_2$.

Then the general solution is $x(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$.

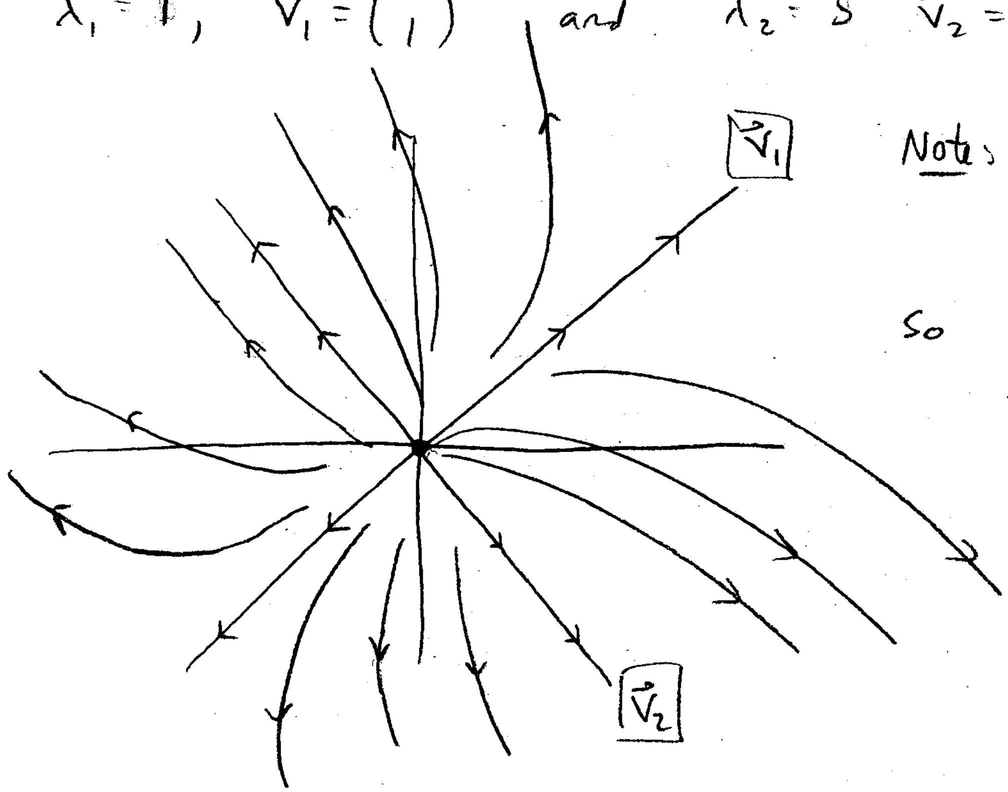
We can plot the "phase portrait" (x_2 vs. x_1) by first drawing the lines on which the eigenvectors lie.

If $\lambda > 0$, then the solutions move away from the origin, because $\lim_{t\to\infty} Ce^{\lambda t}\vec{v} = \pm\infty$.

If $\lambda < 0$, then the solutions move toward the origin, because $\lim_{t\to\infty} Ce^{\lambda t}\vec{v} = 0$.

From this, we can sketch the entire phase portrait.

Example. Sketch the phase portrait for $\vec{x}' = A\vec{x}$, if $\lambda_1 = 1$, $\vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\lambda_2 = 5$ $\vec{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$.



Note: $x_1(t) = C_1 e^{5t}\begin{pmatrix}1\\1\end{pmatrix}$

$x_2(t) = C_2 e^{t}\begin{pmatrix}1\\-1\end{pmatrix}$

So $x_1(t)$ grows "faster" than $x_2(t)$