MTHSC 208, WEEK 9

Week 8 summary:

- We can write a system of two 1st order linear ODEs

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + b_1 & x_1(t_0) = d_1 \\ x_2' = a_{21}x_1 + a_{22}x_2 + b_2 & x_2(t_0) = d_2 \end{cases}$$

as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \text{ or just}$$

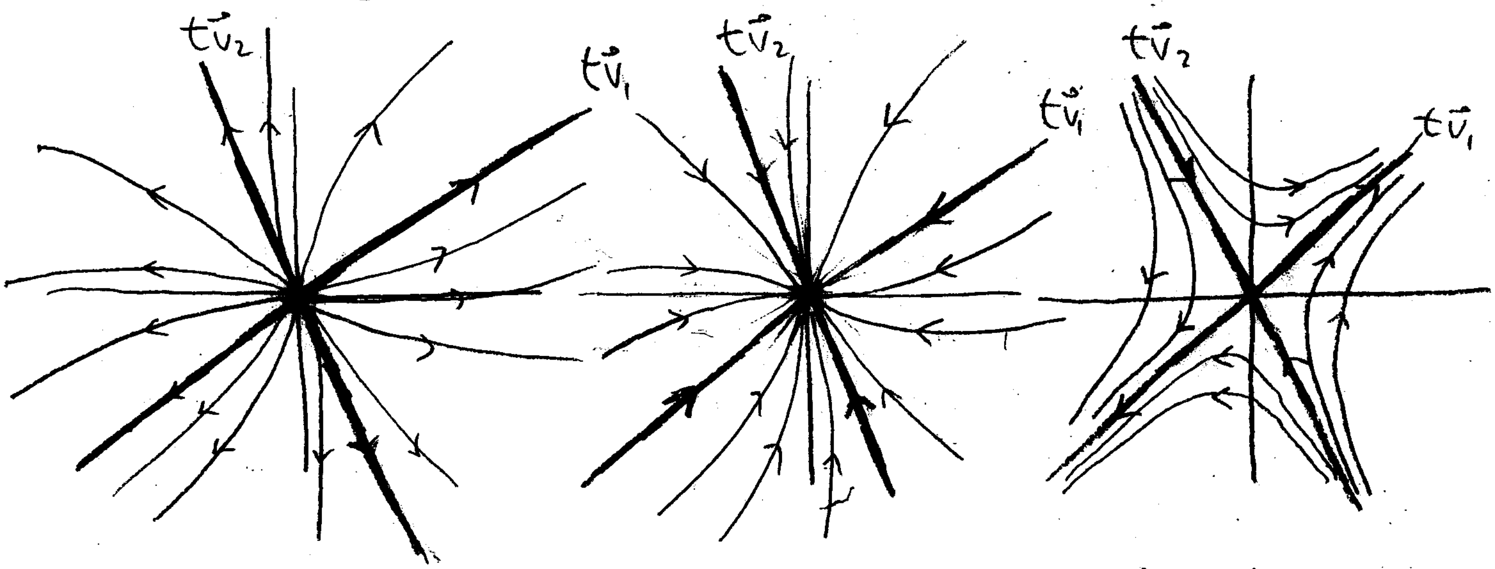
$$\vec{x}' = A\vec{x} + \vec{b}, \quad \vec{x}(t_0) = \vec{d}.$$

- When the eigenvalues are real & distinct, the general solution is

$$\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2.$$

- We can visualize the solutions via a <u>phase plot</u> (x_2 vs. x_1).

<u>Examples</u> (Say $|\lambda_1| > |\lambda_2|$)



$\lambda_1, \lambda_2 > 0$ $\quad$ $\lambda_1, \lambda_2 < 0$ $\quad$ $\lambda_1 > 0, \lambda_2 < 0$

<u>This week:</u> Complex & repeated eigenvalues

<u>Example</u>: (Complex eigenvalues): $\vec{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{x}$

call this A.

Recall that $\lambda_1 = -\frac{1}{2} + i \qquad \lambda_2 = -\frac{1}{2} - i$

$\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$

As before, we have two solutions: $\vec{x}_1(t) = e^{(-\frac{1}{2}+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $\vec{x}_2(t) = e^{(-\frac{1}{2}-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

but we'd prefer to write these using sines & cosines,

Recall Euler's formula: $e^{it} = \cos t + i \sin t$.

$$\vec{x}_1(t) = e^{-\frac{1}{2}t}e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-\frac{1}{2}t}(\cos t + i \sin t)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix}}_{\vec{u}(t)} + i \underbrace{\begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}}_{\vec{w}(t).} = \vec{u}(t) + i\,\vec{w}(t).$$

<u>Check</u>: $\vec{u}(t)$ and $\vec{w}(t)$ both satisfy the ODE

i.e, $\vec{u}' = A\vec{u}$ and $\vec{w}' = A\vec{w}$.

These are 2 "distinct" (not scalar multiples of each other) solutions.

Therefore, the general solution is $\vec{x}(t) = C_1\vec{u}(t) + C_2\vec{w}(t)$.

i.e, $\vec{x}(t) = C_1\begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + C_2\begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}$

Recall the "1D" case, when $r = a + bi$

We had a sol'n $x_1(t) = e^{at}e^{ibt} = e^{at}(\cos bt + i \sin bt) = \underbrace{e^{at}\cos bt}_{u(t)} + i\,\underbrace{e^{at}\sin bt}_{w(t)}$

And the general solution was $x_1(t) = C_1u(t) + C_2w(t) = e^{at}(A\cos bt + B\sin bt)$.

It's really the same thing!

Let's plot this solution:

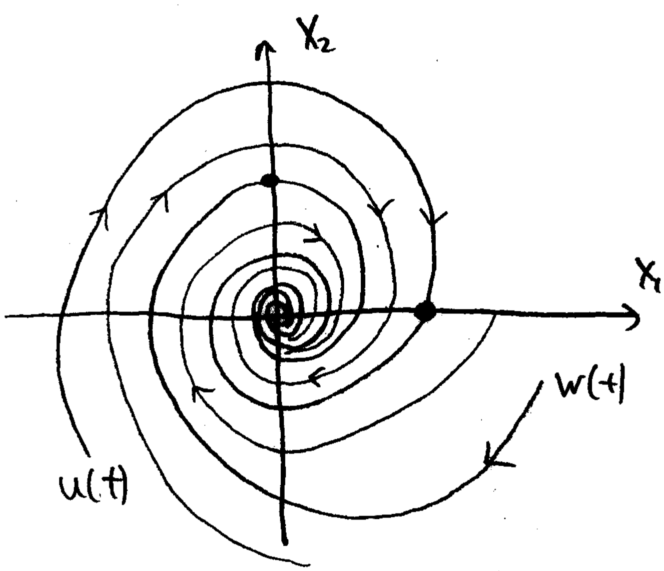
$$\vec{x}(t) = C_1 \begin{pmatrix} e^{-t/2} \cos t \\ -e^{-t/2} \sin t \end{pmatrix} + C_2 \begin{pmatrix} e^{-t/2} \sin t \\ e^{-t/2} \cos t \end{pmatrix} = C_1 e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

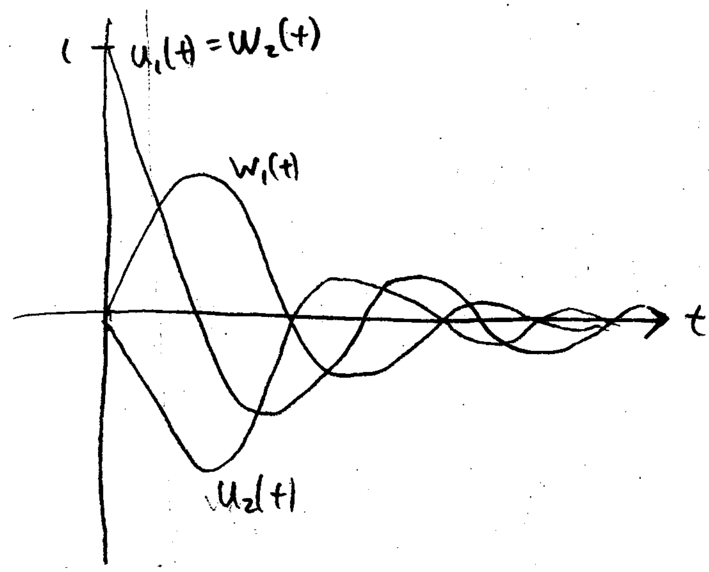
<u>Note:</u> The curves $\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ and $\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ are (clockwise) <u>circles</u>.

(In vector calculus, we wrote these as, e.g., $c(t) = (\cos t, \sin t)$.)

Then, $u(t) = e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ is a <u>decaying spiral</u>, starting at $(1, 0)$.

and $w(t) = e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ is a decaying spiral, starting at $(0, 1)$.





Phase plot of the system $\vec{x}' = A\vec{x}$ Component plots.

<u>Note:</u> This "decayed" because $\lim_{t \to \infty} e^{-\frac{1}{2}t} = 0$

Here, the real part of $\lambda_{1,2} = a + bi$ was $a = -\frac{1}{2} < 0$, and so $\lim_{t \to \infty} e^{at} = 0$.

(If the real part was $a > 0$, then $\lim_{t \to \infty} e^{at} = \infty$, and the spiral would be <u>expanding</u> (but still clockwise).

Summary of method

Suppose we wish to solve $\vec{x}' = A\vec{x}$, and $\lambda_{1,2} = a \pm bi$.

We have 2 solutions: $\vec{x}_1(t) = e^{(a+bi)t}\vec{v}_1$ and $\vec{x}_2(t) = e^{(a-bi)t}\vec{v}_2$.

Take one of these (say $\vec{x}_1(t)$), and write it as $\vec{x}_1(t) = \vec{u}(t) + i\,\vec{w}(t)$

The general solution is $\vec{x}(t) = C_1\vec{u}(t) + C_2\vec{w}(t)$.

The phase plot will be spiraling circles/ellipses (need not be perfect circles!)

decaying if $a < 0$

expanding if $a > 0$

stable if $a = 0$ (see next example).

Example: (Ellipses, purely imaginary eigenvalues, i.e, $a=0$):

Consider the system $\vec{x}' = \begin{pmatrix} 1/2 & -5/4 \\ 2 & -1/2 \end{pmatrix}\vec{x}$, $\vec{x}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

Check: $|A - \lambda I| = \lambda^2 + \frac{9}{4} = 0 \Rightarrow \lambda_1 = \frac{3}{2}i$, $\lambda_2 = -\frac{3}{2}i$,

and $\vec{v}_1 = \begin{pmatrix} 5 \\ 2-6i \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 5 \\ 2+6i \end{pmatrix}$.

We have 2 solutions: $\vec{x}_1(t) = e^{3t/2}\begin{pmatrix} 5 \\ 2-6i \end{pmatrix}$, $\vec{x}_2(t) = e^{-3t/2}\begin{pmatrix} 5 \\ 2+6i \end{pmatrix}$.

Separate $\vec{x}_1(t)$ into real & imaginary parts: $(\vec{x}_1(t) = \vec{u}(t) + i\,\vec{w}(t))$.

$$\vec{x}_1(t) = \left(\cos\tfrac{3}{2}t + i\sin\tfrac{3}{2}t\right)\begin{pmatrix} 5 \\ 2-6i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 5\cos\frac{3}{2}t \\ 2\cos\frac{3}{2}t + 6\sin\frac{3}{2}t \end{pmatrix}}_{\vec{u}(t)} + i\underbrace{\begin{pmatrix} 5\sin\frac{3}{2}t \\ 2\sin\frac{3}{2}t - 6\cos\frac{3}{2}t \end{pmatrix}}_{\vec{w}(t)} = \vec{u}(t) + i\,\vec{w}(t).$$

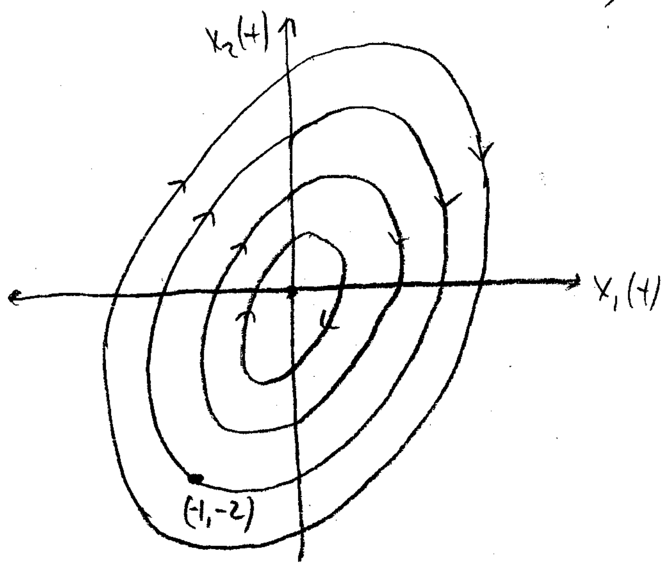
Thus, the general solution is

$$\vec{x}(t) = C_1 \begin{pmatrix} 5 \cos \frac{3}{2}t \\ 2 \cos \frac{3}{2}t + 6 \sin \frac{3}{2}t \end{pmatrix} + C_2 \begin{pmatrix} 5 \sin \frac{3}{2}t \\ 2 \sin \frac{3}{2}t - 6 \cos \frac{3}{2}t \end{pmatrix}$$

Let's plot the curve satisfying $\vec{x}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

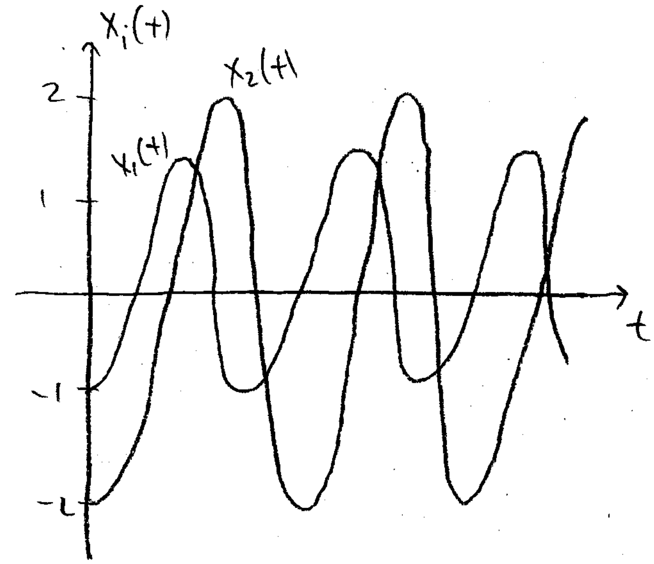
$$\vec{x}(0) = C_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \Rightarrow C_1 = \frac{-1}{5}, \quad C_2 = \frac{4}{15}$$

$$\Rightarrow x_1(t) = -\cos \frac{3}{2}t + \frac{4}{3} \sin \frac{3}{2}t, \qquad x_2(t) = -2 \cos \frac{3}{2}t - \frac{2}{3} \sin \frac{3}{2}t$$



Phase portrait

(Plugged $(x_1(t), x_2(t))$ into WolframAlpha)



Component plots

(Plugged $x_1(t)$ & $x_2(t)$ into WolframAlpha).