MTHSC 208, WEEK 10

Week 9 Summary:

- $\vec{x}' = A\vec{x}$, A has complex eigenvalues.

  We'll summarize this next week (since week 9 was Fall Break).

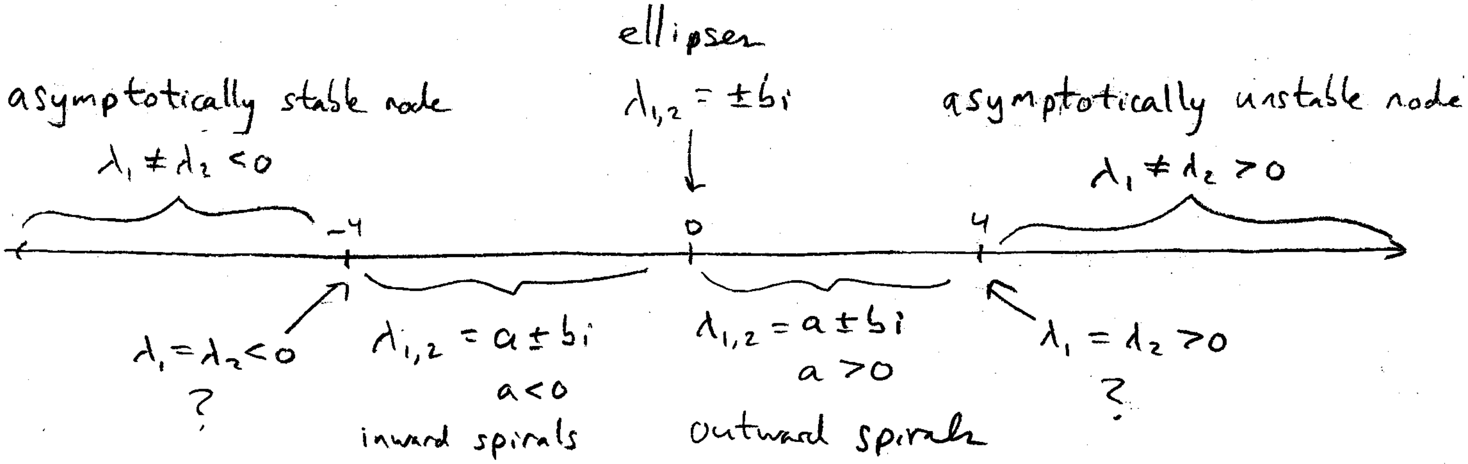
This week: Repeated eigenvalues.

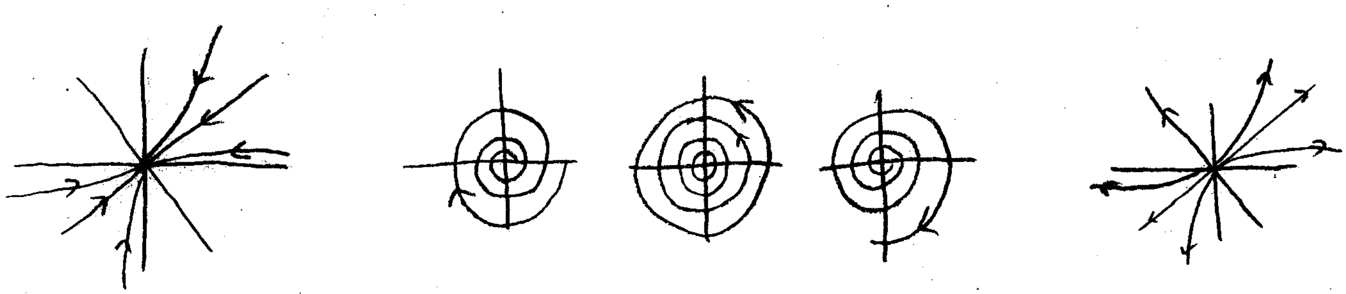
Example: Consider $\vec{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \vec{x}$, where α is some parameter.

Check: $|A - \lambda I| = \lambda^2 - \alpha\lambda + 4 = 0$, so the eigenvalues are

$$\boxed{\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}}$$

Let's see what the phase portrait looks like as α changes.





Comments:

* There are 3 "critical values" of α:

  $\alpha = 0$ (purely imaginary eigenvalue): ellipses

  $\alpha = \pm 4$ (Repeated eigenvalue): we'll analyze this next.

* For this system, there aren't any values of α that yield $\lambda_1 > 0$, $\lambda_2 < 0$. (saddle point).

Example: (Repeated eigenvalues, $\underline{2}$ eigenvectors).

Consider the system $\vec{x}' = A\vec{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \vec{x}$.

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 = 0 \implies \lambda_1 = \lambda_2 = -1.$$

$(A + I)\vec{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $\underline{\text{every}}$ vector is an eigenvector.

Let's pick 2: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Now, we have 2 solutions: $\vec{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{x}_2(t) = e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

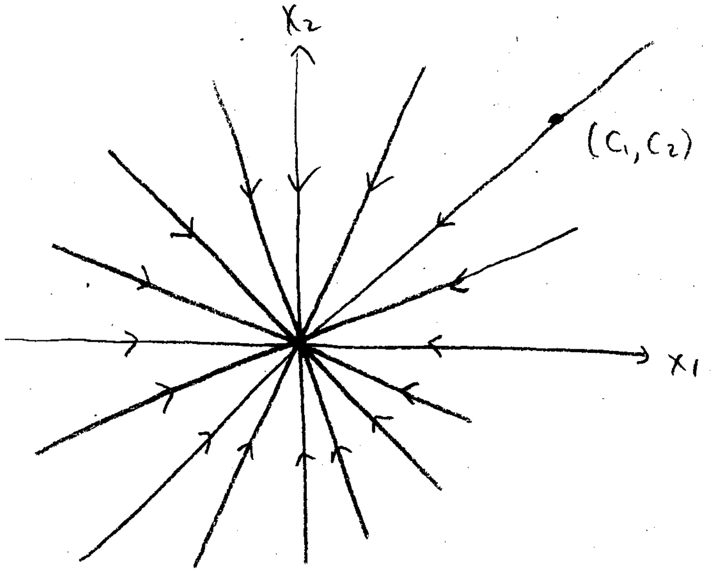
so the general solution is
$$\vec{x}(t) = C_1 \vec{x}_1(t) + C_2 \vec{x}_2(t) = C_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note: We could have solved this "easy" system the "old way."

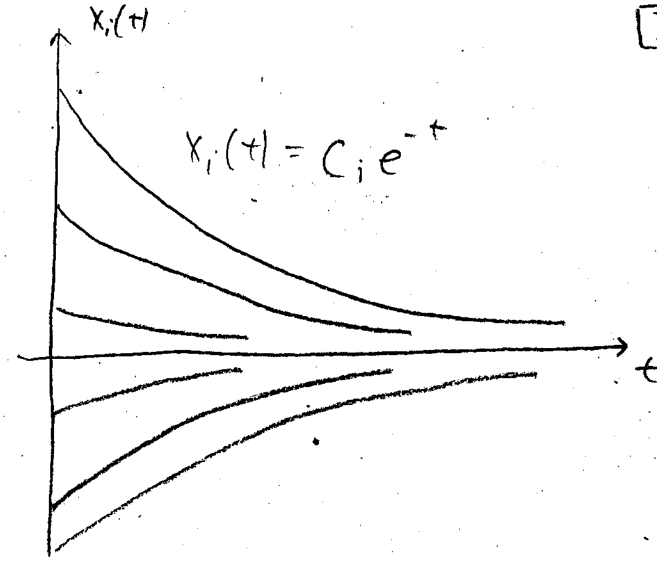
$$\begin{cases} x_1' = -x_1 \\ x_2' = -x_2 \end{cases} \implies \begin{matrix} x_1(t) = C_1 e^{-t} \\ x_2(t) = C_2 e^{-t} \end{matrix}$$

Let's plot this: starting from $\vec{x}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$,

$$\frac{x_2(t)}{x_1(t)} = \frac{C_2 e^{-t}}{C_1 e^{-t}} = \frac{C_2}{C_1} \quad \text{and} \quad \lim_{t\to\infty} X(t) = \lim_{t\to\infty} e^{-t}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$



Phase Portrait



Component plots

* $\vec{0}$ is called a <u>proper node</u>, or <u>star point</u>

<u>Example:</u> (Repeated negative eigenvalues, $\underline{\underline{1}}$ eigenvector)

Consider the system $\vec{x}' = A\vec{x} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \vec{x}$.

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = (1+\lambda)(3+\lambda)^2 + 1 = \lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow \lambda_{1,2} = -2.$$

$$(A + 2I)\vec{v} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

Thus, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the <u>only</u> eigenvector (up to scalar multiple).

We have <u>one</u> solution: $\vec{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

To find a 2^{nd} solution, assume $\boxed{\vec{x}_2(t) = t e^{-2t}\vec{v} + e^{-2t}\vec{w}}$

and solve for $\vec{v}$ & $\vec{w}$.

<u>Note:</u> $\vec{x}_2'(t) = -2te^{-2t}\vec{v} + e^{-2t}(\vec{v} - 2\vec{w})$ (product rule)

Plug back into $\vec{x}' = A\vec{x}$:

$$-2te^{-2t}\vec{v} + e^{-2t}(\vec{v} - 2\vec{w}) = A\left(te^{-2t}\vec{v} + e^{-2t}\vec{w}\right)$$

Now, equate coefficients of $\underline{te^{-2t}}$ and $\underline{e^{-2t}}$:

te^{-2t} : $\quad -2\vec{v} = A\vec{v} \quad \Longrightarrow \quad (A+2I)\vec{v} = \vec{0} \qquad (1)$

te^{-2t} : $\quad \vec{v} - 2\vec{w} = A\vec{w}. \quad \Longrightarrow \quad (A+2I)\vec{w} = \vec{v} \qquad (2)$

<u>Note:</u> The solution to (1) is the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of A.

Plug this back into (2): $\quad (A+2I)$

$$(A+2I)\vec{w} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} w_1 - w_2 = 1 \\ w_1 - w_2 = 1 \end{cases}$$

$\Rightarrow \boxed{w_1 - w_2 = 1}$ If $w_1 = c$, then $w_2 = -1+c$

$$\Rightarrow \vec{w} = \begin{pmatrix} c \\ -1+c \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Plug this back into $\vec{x}_2(t) = te^{-2t}\vec{v} + e^{-2t}\vec{w}$

$$\vec{x}_2(t) = te^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \underbrace{ce^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{multiple of } \vec{x}_1(t), \text{ so we may ignore this.}}$$

$$\Rightarrow \vec{x}_2(t) = te^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Thus, our general solution is

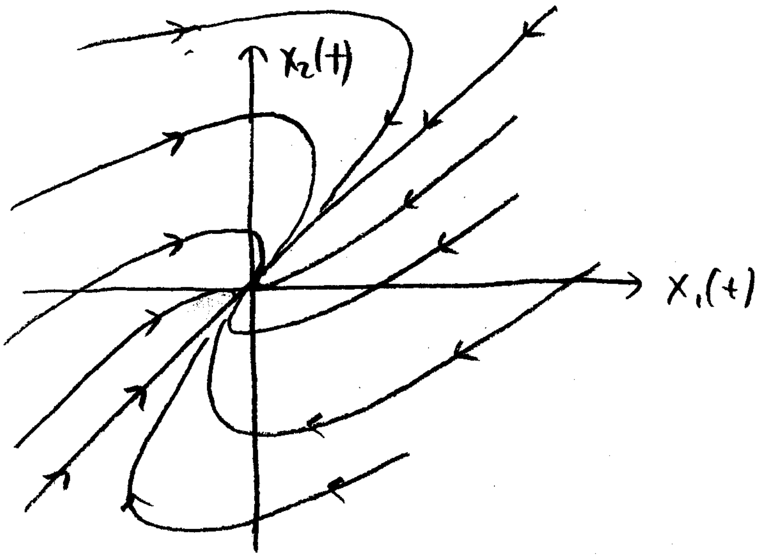
$$\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) = \boxed{c_1e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-2t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]}$$

<u>Analyze long-term behavior:</u>

<u>Note:</u> $\lim_{t\to\infty} x(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Moreover, $\quad x(t) = \underbrace{c_1e^{-2t}}_{\to 0 \text{ faster}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \underbrace{c_2e^{-2t}}_{\to 0 \text{ faster}}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \underbrace{c_2te^{-2t}}_{\to 0 \text{ slower}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus, the $te^{-2t}\begin{pmatrix}1\\1\end{pmatrix}$ term "dominates" in the limit.



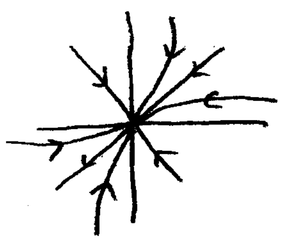
Phase portrait.

* As before, we get a solution lying on the "eigenvector line,"
$t\vec{v}_1 = t\begin{pmatrix}1\\1\end{pmatrix}$.

* The point $\vec{0}$ is called an <u>improper</u>, or <u>degenerate</u> node.

<u>Note</u>: $\lambda_{1,2} = -2$ is the "critical threshold" between

$\lambda_1 \neq \lambda_2 < 0$ (stable node) and $\lambda_{1,2} = a \pm bi$, $a < 0$ (inward spiral).



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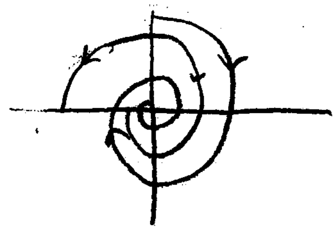


<u>Summary</u> of Phase portraits / stability of $\vec{x}' = A\vec{x}$, $\det A \neq 0$.

<u>Eigenvalues</u>	<u>Phase portrait summary</u>	<u>Type of critical pt</u>
$\lambda_1 > \lambda_2 > 0$	Asymptotically unstable.	Node
$\lambda_1 < \lambda_2 < 0$	Asymptotically stable.	Node
$\lambda_2 < 0 < \lambda_1$	Unstable	Saddle pt.
$\lambda_1 = \lambda_2 > 0$	Unstable	Proper/improper node
$\lambda_1 = \lambda_2 < 0$	Asymptotically stable	Proper/improper node
$\lambda_{1,2} = a \pm bi$	Spirals / ellipses	Spiral point
$a > 0$	Unstable	Outward
$a < 0$	Asymptotically stable	inward
$a = 0$	Stable.	center

A word on higher-order systems of ODEs

Basically, the same ideas carry over, but the math is more complicated for $n \times n$ matrices, $n > 2$.

SIR model (A popular 3×3 system of ODEs).

This models an epidemic disease in a population (e.g. flu).

Let $S(t)$ = # of susceptible people at time t.

$I(t)$ = # of infected people at time t.

$R(t)$ = # of recovered (immune) people at time t.

Initially, there are N susceptible (uninfected people).

People transition: Susceptible $\longrightarrow$ Infected $\longrightarrow$ Recovered.

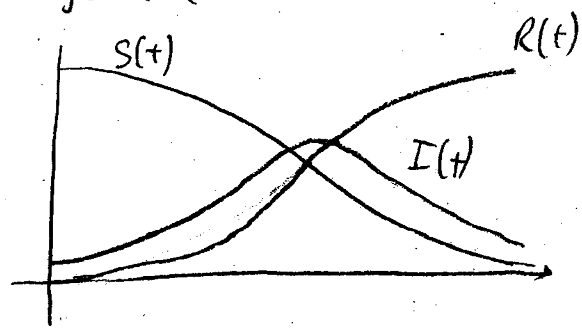
$$\frac{dS}{dt} = -aSI \qquad \text{"proportional to both } S(t) \text{ \& } I(t)\text{"}$$

$$\frac{dI}{dt} = aSI - bI \qquad \text{"(rate people get sick) - (rate people get healthy)"}$$

$$\frac{dR}{dt} = bI \qquad \text{"rate people get healthy"}$$

We get the nonlinear, autonomous system

$$\begin{cases} S' = -aSI & S(0) = N \\ I' = aSI - bI & I(0) = 1 \\ R' = bI & R(0) = 0 \end{cases}$$

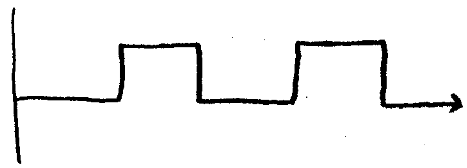


Component plots

Laplace Transforms

- Used to solve linear ODEs.
- Useful when forcing term is discontinuous

  e.g, step function



  Think: Force being turned on/off.

Big idea:

$$\begin{tikzcd}
\text{ODE} \arrow[rrr, "\mathcal{L}"] \arrow[d, dashed] & & & \text{Algebraic equation} \arrow[d] \\
\text{Sol'n to ODE} & & & \text{Sol'n to algebraic eq'n} \arrow[lll, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

The Laplace transform is an operator: it inputs a function, and outputs a function.

Def: Suppose $f(t)$ is defined for $0<t<\infty$. The Laplace transform of f is the function $\mathcal{L}(f)$, where

$$\boxed{\mathcal{L}\{f(t)\}(s) := F(s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s>0}$$

Often, we denote $\mathcal{L}(f)$ by F, i.e, $f \xmapsto{\mathcal{L}} F$

Recall: $\int_0^\infty f(t)\,dt = \lim_{T\to\infty} \int_0^T f(t)\,dt.$

Example: Compute $\mathcal{L}(F)$, where $f(t) = e^{at}$.

$$F(s) = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \frac{e^{-(s-a)}}{-(s-a)}\Big|_0^\infty$$

$$= \underbrace{\lim_{T\to\infty} \frac{e^{-(s-a)}}{-(s-a)}}_{} + \frac{1}{s-a}$$

$$= \begin{cases} 0 & \text{if } s > a \\ \infty & \text{if } s \leq a \end{cases}$$ (i.e, the limit does not exist).

Thus, $\boxed{\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}}$ if $s > a$ (and isn't defined otherwise).

Note: Sometimes the domain is restricted.

e.g., $f(t)$ has domain $(-\infty, \infty)$

$F(s)$ has domain (a, ∞)

(Analogy: e^x has domain $(-\infty, \infty)$ but it's "inverse function" $\ln x$ has domain $\ln x$)

Recall: Integration by parts.

Let's rederive it: $(uv)' = u\,dv + du\, v$

$$u\,dv = (uv)' - v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: Let $f(t) = t$. Compute $\mathcal{L}(f)$.

$$F(s) = \int_0^\infty t e^{-st}\, dt$$

Let $u = t$, $du = dt$, $v = -\frac{1}{s} e^{-st}$, $dv = e^{-st}\, dt$

$$\int \underbrace{t}_{u} \underbrace{e^{-st}\, dt}_{dv} = \underbrace{-\frac{1}{s} t e^{-st}}_{uv} + \underbrace{\frac{1}{s}\int e^{-st}\, dt}_{-\int v\,du} = \frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}$$

$$\mathcal{L}(f) = F(s) = \int_0^\infty t\, e^{-st}\, dt = \lim_{T\to\infty} \int_0^T t\, e^{-st}\, dt$$

$$= \lim_{T\to\infty} \left(\frac{-t e^{-st}}{s} - \frac{e^{-st}}{s^2} \right)\Big|_0^T = \lim_{T\to\infty} \left(\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right) - \left(0 - \frac{e^0}{s^2} \right) = \boxed{\frac{1}{s^2}}$$

Other common functions:

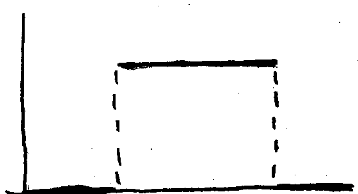
$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\sin at)(s) = \frac{a}{s^2+a^2}$$

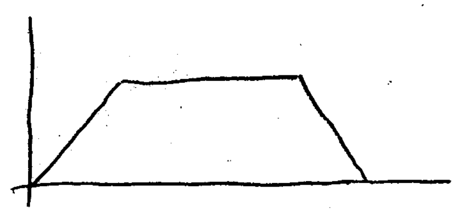
$$\mathcal{L}(\cos at)(s) = \frac{s}{s^2+a^2}$$

We can also compute the Laplace transform of piecewise continuous & piecewise differentiable functions.

e.g.,



step function
piecewise continuous

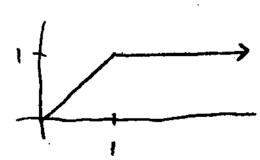


continuous
piecewise differentiable.

Example: let $f(x) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$

Compute $\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^1 = \boxed{\frac{-e^{-s}}{s} + \frac{1}{s}}$

Example: let $f(x) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 < t < \infty \end{cases}$



We must break the integral into 2 parts:

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt = \underbrace{\int_0^1 t\, e^{-st}\, dt}_{I1} + \underbrace{\int_1^\infty e^{-st}\, dt}_{I2}$$

$$I1 = -\frac{e^{-s}}{s} - \left(\frac{e^{-s}}{s^2} - \frac{1}{s^2}\right) \qquad I2 = \lim_{T\to\infty} -\frac{e^{-st}}{s}\Big|_{t=1}^{T} = \frac{e^{-s}}{s}$$

$$F(s) = \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{1}{s^2}\right) + \left(\frac{e^{-s}}{s}\right) = \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2}}$$

Properties of the Laplace Transform:

- $\mathcal{L}$ is linear
- $\mathcal{L}$ turns derivatives into multiplication.

(i) Linearity: $\mathcal{L}(a\,f(t) + b\,g(t))(s) = a\,\mathcal{L}(f(t))(s) + b\,\mathcal{L}(g(t))(s)$.

i.e, you can break apart sums & pull out constants
($\mathcal{L}$ is a "linear operator").

(ii) Turns derivatives into multiplication: $\boxed{\mathcal{L}(y'(t))(s) = s\,Y(s) - y(0)}$

Proof: $\mathcal{L}(y')(s) = \int_0^\infty y'(t)e^{-st}\,dt = \lim_{T\to\infty}\Big[e^{-st}y(t) + \underbrace{s\int_0^T y(t)\,e^{-st}\,dt}_{s\mathcal{L}(y)(s)}\Big]$

$= \lim_{T\to\infty} e^{-st}y(t)\Big|_0^T + s\mathcal{L}(y)(s)$

$= \underbrace{\lim_{T\to\infty} e^{-sT}y(T)}_{\to 0 \text{ as long as } |y(t)| \le Ce^{at}} - y(0) + s\mathcal{L}(y(s)) = s\mathcal{L}(y)(s) - y(0) = s\,Y(s) - y(0)$. ✓

Henceforth, we will make this assumption.

Similarly, $\mathcal{L}(y'')(s) = s^2\,Y(s) - s\,y(0) - y'(0)$

$\mathcal{L}(y''')(s) = s^3\,Y(s) - s^2 y(0) - s\,y'(0) - y''(0)$

$\mathcal{L}(y^{(4)})(s) = s^4\,Y(s) - s^3 y'(0) - s^2 y''(0) - s\,y'''(0)$.

$\vdots$

etc.

Proof: $\mathcal{L}(y'') = s\,\mathcal{L}(y')(s) - y'(0)$

$$= s\left(s\,\mathcal{L}(y)(s) - y(0)\right) - y'(0)$$

$$= s^2 Y(s) - s\,y(0) - y'(0). \checkmark$$

The formula for higher derivatives is handled similarly. $\square$

Application: Consider $y'' - y = e^{2t}$ $\quad y(0)=0, \quad y'(0)=1.$

$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(e^{2t})$$

$$\left[s^2 Y(s) - s\,\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,1}\right] - \left[Y(s)\right] = \frac{1}{s-2}$$

$$s^2 Y - 1 - Y = \frac{1}{s-2} \implies (s^2-1)Y = \frac{1}{s-2} + \frac{s-2}{s-2} = \frac{s-1}{s-2}$$

$$\boxed{Y(s) = \frac{1}{(s+1)(s-2)}}$$

* The solution to the IVP (above) is the function whose Laplace transform is $Y(s) = \frac{1}{(s+1)(s-2)}$ (we'll show how to find this later).

$$\begin{tikzcd}
y'' - y = e^{2t},\ y(0)=0,\ y'(0)=1 \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & s^2 Y - 1 - Y = \frac{1}{s-2} \arrow[d, "\text{solve!}"] \\
\text{solution } y(t) & Y(s) = \frac{1}{(s+1)(s-2)} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

More Laplace transform properties:

(i) $\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c)$

(ii) $\mathcal{L}\{t\, f(t)\}(s) = -F'(s)$

(iii) $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^n(s)$.

Applications of these

Ex 1: $f(t) = e^{2t}\cos 3t$.

Recall: $\mathcal{L}(\cos 3t) = \frac{s}{s^2+9}$

Using (i), $\mathcal{L}(e^{2t}\cos 3t) = \boxed{\frac{s-2}{(s-2)^2+9}} = F(s)$.

Ex 2: $f(t) = t^2 e^{3t}$ let $g(t) = e^{3t}$

Recall: $\mathcal{L}(e^{3t}) = \frac{1}{s-3} = G(s)$.

Using (iii), $\mathcal{L}(t^2 e^{3t})(s) = (-1)^2 F''(s) = 1 \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) = \boxed{\frac{2}{(s-3)^3}}$