MTHSC 208, WEEK 11

Week 10 summary:

- $\vec{x}' = A\vec{x}$

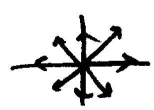
  * A has complex eigenvalues: $\lambda_{1,2} = a \pm bi$

    $\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t)$, where $\vec{x}_1(t) = e^{(a+bi)t}\vec{v} = e^{at}$
    $= e^{at}(\cos bt + i \sin bt)\vec{v}$.

    Write $\vec{x}_1(t)$ as $\vec{u}(t) + i\vec{w}(t)$, and the general solution is
    $\vec{x}(t) = C_1\vec{u}(t) + C_2\vec{w}(t)$.

    Phase portrait: $a > 0$ outward spirals
    $a < 0$ inward spirals
    $a = 0$ ellipses / circles.

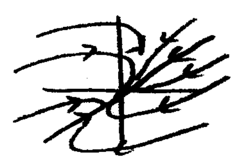
  * A has repeated eigenvalues:

    (i) Two eigenvectors: star point.  or 

    $\vec{x}(t) = C_1e^{\lambda t}\vec{v}_1 + C_2e^{\lambda t}\vec{v}_2$

    (ii) One eigenvector: $\vec{x}_1(t) = C_1e^{\lambda t}\vec{v}$.

    Guess: $\vec{x}_2(t) = te^{\lambda t}\vec{u} + e^{\lambda t}\vec{w}$, solve for $\vec{u}$ & $\vec{w}$.

    Phase portrait: Degenerate node: 

- Laplace transforms: $\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt := F(s)$
  * Turns derivatives into multiplication: $\mathcal{L}(y') = sY - y(0)$
  * Useful especially when forcing term is discontinuous:

$$\begin{array}{ccc} \text{ODE} & \xrightarrow{\mathcal{L}} & \text{Algebraic equation} \\ \downarrow & & \downarrow \\ \text{sol'n to ODE} & \xleftarrow{\mathcal{L}^{-1}} & \text{Solution to alg. eq'n} \end{array}$$

More Laplace transform properties

(i) $\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c)$

(ii) $\mathcal{L}\{t\, f(t)\}(s) = -F'(s)$

(iii) $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$.

Applications of these:

Example 1: $f(t) = e^{2t} \cos 3t$

Recall: $\mathcal{L}(\cos 3t) = \frac{s}{s^2+9}$

Using (i), $\mathcal{L}(e^{2t} \cos 3t) = \boxed{\frac{s-2}{(s-2)^2+9}} = F(s)$

Example 2: $f(t) = t^2 e^{3t}$ Let $g(t) = e^{3t}$

Recall: $\mathcal{L}(e^{3t}) = \frac{1}{s-3} = G(s)$

Using (iii), $\mathcal{L}(t^2 e^{3t})(s) = (-1)^2 G''(s) = 1 \frac{d}{ds}\left(\frac{1}{s-3}\right) = \boxed{\frac{2}{(s-3)^3}}$

Back to using Laplace transforms to solve ODEs:

Example: $y' - 4y = \cos 2t, \quad y(0) = -2$

$\mathcal{L}(y') - \mathcal{L}(4y) = \mathcal{L}(\cos 2t)$

$[sY - y(0)] - 4Y = \frac{s}{s^2+4}$

$sY + 2 - 4Y = \frac{s}{s^2+4} \Rightarrow (s-4)Y = \frac{s}{s^2+4} - \frac{2(s^2+4)}{s^2+4} = \frac{-2s^2+s-8}{s^2+4}$

$\Rightarrow Y(s) = \frac{-2s^2+s-8}{(s-4)(s^2+4)}$

* The solution to this IVP is the <u>unique</u> function $y(t)$ that has Laplace transform $Y(s) = \frac{-2s^2+s-8}{(s-4)(s^2+4)}$

Now, we need to figure out how to "undo" the Laplace transform.

Let's revisit our example IVP: $y'' - y = e^{2t}$, $y(0) = 0$, $y'(0) = 1$.

Recall that we computed $Y(s) = \frac{1}{(s+1)(s-2)}$.

To solve for $y(t)$, we must compute $\mathcal{L}^{-1}\left(\frac{1}{(s+1)(s-2)}\right)$.

To do this, write $\frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$.

Use partial fraction decomposition:

$$\frac{A(s-2)}{(s+1)(s-2)} + \frac{B(s+1)}{(s-2)(s+1)} = \frac{1}{(s+1)(s-2)} \Longrightarrow \overbrace{(A+B)}^{=0}s + \overbrace{(B-2A)}^{=1} = 1$$

$$\Rightarrow \begin{cases} A+B=0 \\ B-2A=1 \end{cases} \Rightarrow A=-B \Rightarrow 3B=1 \Rightarrow \begin{matrix} B = 1/3 \\ A = -1/3. \end{matrix}$$

So, $\frac{1}{(s+1)(s-2)} = \frac{-1/3}{s+1} + \frac{1/3}{s-2}$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s-2)}\right) = \mathcal{L}^{-1}\left(\frac{-1/3}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1/3}{s-2}\right)$$

$$= -\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3}\mathcal{L}\left(\frac{1}{s-2}\right) = \boxed{-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}}$$

Example: Compute $\mathcal{L}^{-1}\left(\frac{1}{s^2+4s+13}\right)$.

Note: $s^2+4s+13$ doesn't factor.

Instead, put it in the form $\frac{1}{(s-b)^2+a^2}$, because

$$\mathcal{L}(e^{bt}\sin at) = \frac{a}{(s-b)^2+a^2}$$

"Complete the square": $\frac{1}{(s^2+4s+4)+9} = \frac{1}{(s+2)^2+3^2} = \frac{1}{3}\frac{1}{(s+2)^2+3^2} \xmapsto{\mathcal{L}^{-1}} \boxed{\frac{1}{3}e^{-2t}\sin 3t}$

Example: Solve the IVP $y''-2y'-3y=0, \quad y(0)=1, \quad y'(0)=0.$

Old method: $y(t)=e^{rt}, \quad e^{rt}(r^2-2r-3)=0$

$$\Rightarrow (r-3)(r+1)=0$$

$$\Rightarrow y(t)=C_1e^{3t}+C_2e^{-t}$$

Use initial conditions: $y(0)=C_1+C_2=1$ and $y'(t)=3C_1e^{3t}-C_2e^{-t}$

$$y'(0)=3C_1-C_2=0.$$

$$\Rightarrow \begin{cases} C_1+C_2=1 \\ 3C_1-C_2=0 \end{cases} \Rightarrow C_1=\tfrac{1}{4}, \quad C_2=\tfrac{3}{4} \Rightarrow \boxed{y(t)=\tfrac{1}{4}e^{3t}+\tfrac{3}{4}e^{-t}}$$

New method: $y''-2y'-3y=0, \quad y(0)=1, \quad y'(0)=0.$

$$\mathcal{L}(y'')-2\mathcal{L}(y')-3\mathcal{L}(y)=0.$$

$$[s^2Y-sy(0)-y'(0)]-2[sY-y(0)]-3Y=0$$

$$[s^2Y-s-0]-2[sY-1]-3Y=0 \Rightarrow (s^2-2s-3)Y=s-2$$

$$Y=\frac{s-2}{s^2-2s-3}=\frac{A}{s-3}+\frac{B}{s+1}=\frac{s-2}{s^2-2s-3}$$

$$\frac{A}{s-3}\frac{(s+1)}{(s+1)}+\frac{B}{s+1}\frac{(s-3)}{(s-3)}=\frac{(A+B)s+(A-3B)}{(s+1)(s-3)}=\frac{s-2}{(s+1)(s-3)}$$

$$\begin{cases} A+B=1 \\ A-3B=-2 \end{cases} \Rightarrow \begin{matrix} A=\tfrac{1}{4} \\ B=\tfrac{3}{4} \end{matrix} \Rightarrow Y(s)=\frac{1/4}{s-3}+\frac{3/4}{s+1}$$

$$y(t)=\mathcal{L}^{-1}\left(\frac{1/4}{s-3}\right)+\mathcal{L}^{-1}\left(\frac{3/4}{s+1}\right)=\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)+\frac{3}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow \boxed{y(t)=\tfrac{1}{4}e^{3t}+\tfrac{3}{4}e^{-t}}$$

<u>Summary/analysis</u>: Consider $ay''+by'+cy=f(t)$, $y(0)=y_0$, $y'(0)=y_1$

$$\mathcal{L}(ay''+by'+cy) = a\mathcal{L}(y'')+b\mathcal{L}(y')+c\mathcal{L}(y)$$
$$= a(s^2Y - sy(0) - y'(0)) + b(sY - y(0)) + cY$$
$$= (as^2+bs+c)Y - y_0(as+b) - ay_1 = F(s).$$

Thus, $$Y(s) = \underbrace{\frac{F(s)}{as^2+bs+c}}_{\text{"state-free" sol'n}} + \underbrace{\frac{y_0(as+b)+ay_1}{as^2+bs+c}}_{\text{"input-free" sol'n}}$$

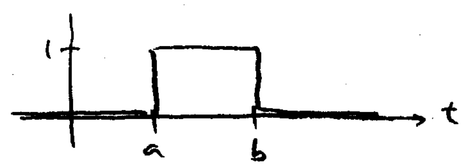
i.e, $Y(s) = Y_s(s) + Y_i(s)$, where

$Y_s(s)$ doesn't depend on the initial conditions, and

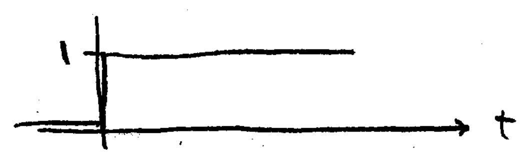
$Y_i(s)$ doesn't depend on the forcing term $f(t)$.

<u>Discontinuous Forcing terms</u>

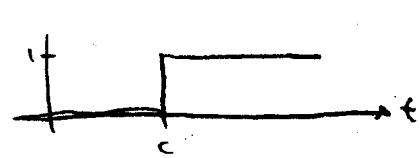
- Interval Function: $H_{ab}(t) = \begin{cases} 0 & t<a \\ 1 & a\le t<b \\ 0 & a\le t<\infty \end{cases}$



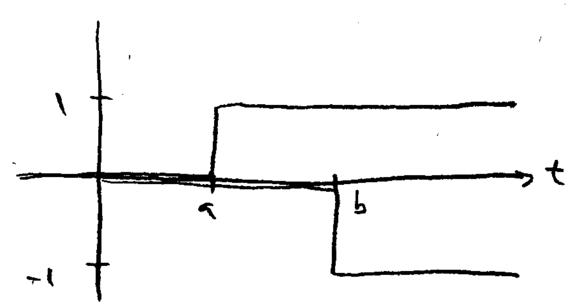
- Heavyside function: $H(t) = \begin{cases} 0 & t<0 \\ 1 & t\ge 0 \end{cases}$



- Shifted Heavyside function: $H_c(t) = \begin{cases} 0 & t<c \\ 1 & t\ge c \end{cases}$
$= H(t-c)$

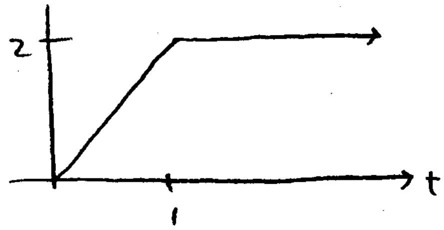


<u>Note</u>: $H_{ab}(t) = H_a(t) - H_b(t) = H(t-a) - H(t-b)$



* Many piecewise continuous functions can be represented using Heavyside functions.

Ex:

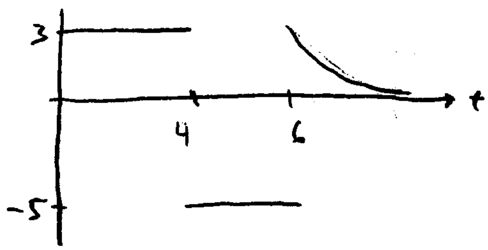


$$f(t) = \begin{cases} 2t & 0 \le t \le 1 \\ 2 & t \geqslant 1 \end{cases}$$

$$f(t) = 2t\, H_{01}(t) + 2 H_1(t)$$
$$= 2t\left[H(t) - H(t-1)\right] + 2H(t-1)$$
$$= 2t\, H(t) - 2(t-1)H(t-1).$$

Goal: Use the Heavyside function to write discontinuous functions, so we can take their Laplace transforms easily.

Example: $f(t) = \begin{cases} 3 & 0 \le t < 4 \\ -5 & 4 \le t < 6 \\ e^{7-t} & 6 \le t < \infty \end{cases}$



$$f(t) = 3H_{04}(t) - 5H_{46}(t) + e^{7-t}H_6(t)$$
$$= 3\left[H(t) - H(t-4)\right] - 5\left[H(t-4) - H(t-6)\right] + e^{7-t}\left[H(t-6)\right]$$
$$= 3H(t) - 8H(t-4) + 5H(t-6) + e^{7-t}H(t-6)$$

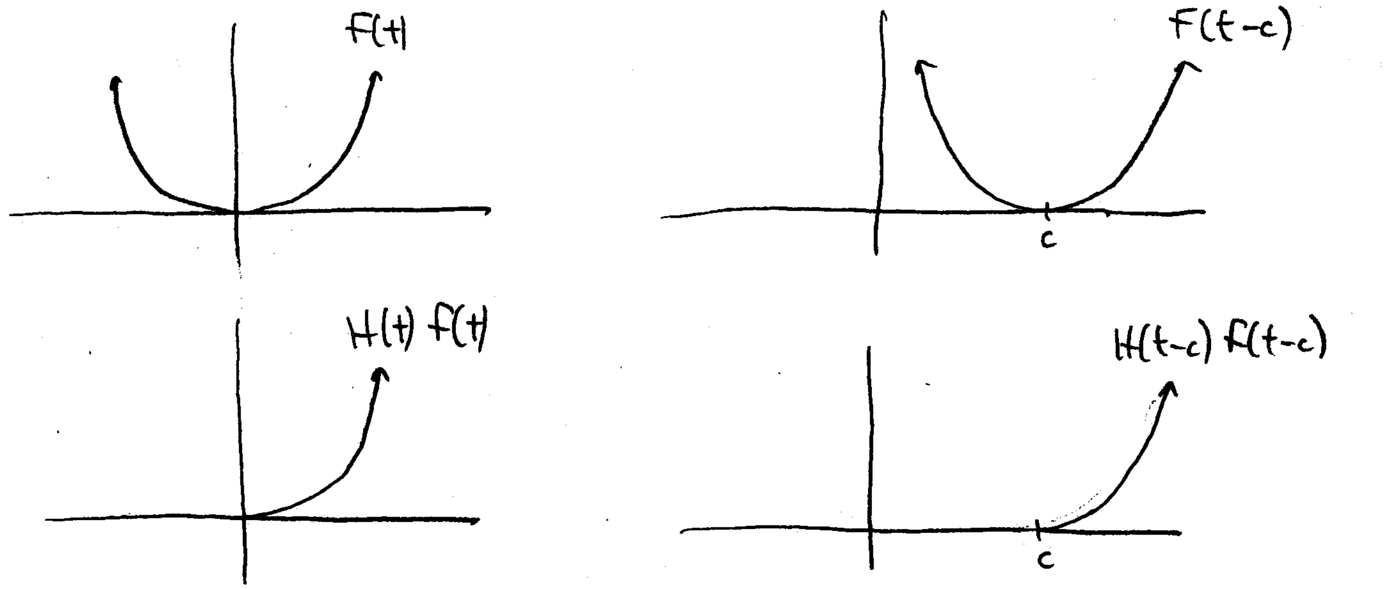
We'll soon see how to quickly take the Laplace transform of this.

Note: $\mathcal{L}\{H_{ab}(t)\}(s) = \int_a^b e^{-st}\,dt = \dfrac{e^{-as} - e^{-bs}}{s}.$

Recall that $\mathcal{L}\{e^{ct}f(t)\}(s) = F(s-c).$

Prop: $\boxed{\mathcal{L}\{H(t-c)\,f(t-c)\}(s) = e^{-cs}F(s)}$

What does this mean? Suppose $f(t) = t^2$

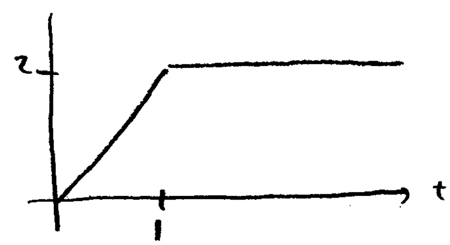


Thus, $H(t-c)f(t-c)$ truncates $f(t)$ at $t=0$, then shifts it by c.

Since $f(t)$ is defined for only $t \geq 0$, we don't want $f(t-c)$ to be defined for $t < c$, so we use $H(t-c)f(t-c)$.

Example: Find the Laplace transform of $g(t) = \begin{cases} 2t & 0 \leq t < 1 \\ 2 & 1 \leq t < \infty \end{cases}$

$$\begin{aligned} g(t) &= 2t\, H_{01}(t) + 2H_1(t) \\ &= 2t\left[H(t) - H(t-1)\right] + 2\left[H(t-1)\right] \\ &= 2t\, H(t) + (2-2t)H(t-1) \\ &= 2t\, H(t) - 2(t-1)H(t-1) \end{aligned}$$



$$\mathcal{L}\{g\} = 2\mathcal{L}\{tH(t)\} - 2\mathcal{L}\{(t-1)H(t-1)\} = \frac{2}{s^2} - \frac{2}{s^2}e^{-s}$$