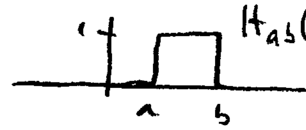
MTHSC 208, WEEK 12

Week 11 summary:

- Piecewise continuous functions can be written concisely using Heavyside functions:



$H_{ab}(t) = H(t-a) - H(t-b)$.

- $\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c)$

  $\mathcal{L}\{f(t-c)\, H(t-c)\}(s) = e^{-cs} F(s)$ (if $c \geq 0$).

More practice:

- $\mathcal{L}\{(t-3)^2 H(t-3)\}(s) = e^{-3s} F(s)$

  $f(t-3) = (t-3)^2$, so $f(t) = f((t+3)-3) = ((t+3)-3)^2 = t^2$

  $F(s) = \frac{1}{s^2} \Rightarrow \mathcal{L}\{(t-3)^2 H(t-3)\}(s) = e^{-3s} F(s) = \boxed{\frac{e^{-3s}}{s^2}}$

- $\mathcal{L}\{t^2 H(t-3)\}(s) = e^{-3s} F(s)$

  $f(t-3) = t^2$, so $f(t) = f((t+3)-3) = (t+3)^2 = t^2 + 6t + 9$

  $F(s) = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \Rightarrow \mathcal{L}\{t^2 H(t-3)\}(s) = e^{-3s} F(s) = \boxed{e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)}$

- $\mathcal{L}\{e^{t-1} H(t-1)\}(s) = e^{-s} F(s)$.

  $f(t-1) = e^{t-1}$, so $f(t) = f((t+1)-1) = e^{(t+1)-1} = e^t$.

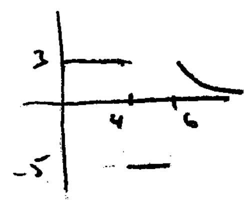
  $F(s) = \frac{1}{s-1} \Rightarrow \mathcal{L}\{e^{t-1} H(t-1)\}(s) = \boxed{e^{-s} \cdot \frac{1}{s-1}}$

- $\mathcal{L}\{e^{7-t}H(t-6)\}(s) = e^{-6s}F(s)$

$f(t-6) = e^{7-t}$, so $f(t) = f((t+6)-6) = e^{7-(t+6)} = e^{1-t} = e \cdot e^{-t}$

$F(s) = e \cdot \frac{1}{s+1} \Rightarrow \mathcal{L}\{e^{7-t}H(t-6)\}(s) = e^{-6s}F(s) = e^{-6s} \cdot e \frac{1}{s+1} = \boxed{\frac{e^{1-6s}}{s+1}}$

Example: Find $F(s)$, where $f(t) = \begin{cases} 3 & 0 \le t < 4 \\ -5 & 4 \le t < 6 \\ e^{7-t} & 6 \le t < \infty \end{cases}$



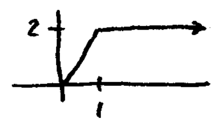
Last week: $f(t) = 3H(t) - 8H(t-4) + 5H(t-6) + e^{7-t}H(t-6)$

$$F(s) = \frac{3}{s} - \frac{8}{s}e^{-4s} + \frac{5}{s}e^{-6s} + \frac{1}{s+1}e^{1-6s}$$

$$= \frac{3 - 8e^{-4s} + 5e^{-6s}}{s} + \frac{e^{1-6s}}{s+1}$$

Example: Solve the IVP: $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$,

where $f(t) = \begin{cases} 2t & 0 \le t \le 1 \\ 2 & t \ge 1 \end{cases}$



Last week: $f(t) = 2t\,H(t) - 2(t-1)H(t-1)$

$$F(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2}$$

Take $\mathcal{L}$ of both sides of the ODE:

$$[s^2Y - s\,y(0) - y'(0)] + Y = \frac{2 - 2e^{-s}}{s^2}$$

$$s^2Y - 1 + Y = \frac{2 - 2e^{-s}}{s^2}$$

$$(s^2+1)Y = \frac{2 - 2e^{-s}}{s^2} + 1 \Rightarrow Y(s) = \frac{2 - 2e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2+1}$$

Partial fractions: $\dfrac{1}{s^2(s^2+1)} = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$

So, $Y(s) = \dfrac{2}{s^2} - \dfrac{2}{s^2+1} - \dfrac{2e^{-s}}{s^2} + \dfrac{2e^{-s}}{s^2+1} + \dfrac{1}{s^2+1}$

$$= \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{1}{s^2+1} + \frac{2e^{-s}}{s^2+1}$$

$$y(t) = 2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+1}\right)$$

$$= 2t - 2(t-1)H(t-1) - \sin t + 2\sin(t-1)H(t-1)$$

$$= [2t - \sin t] + [2\sin(t-1) - 2(t-1)]H(t-1)$$

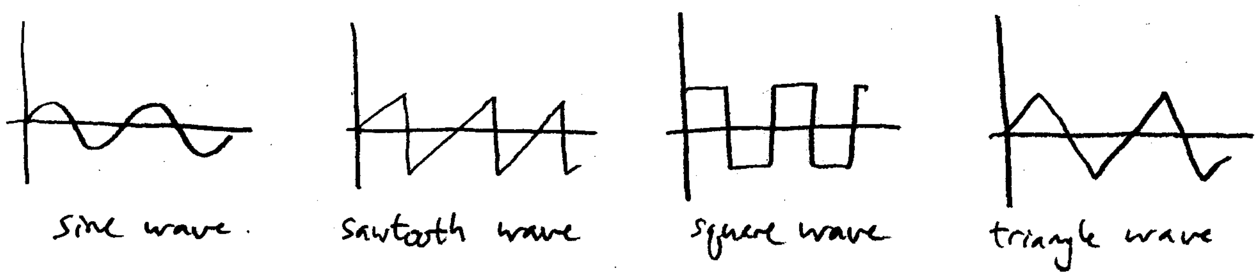
$$= \begin{cases} 2t - \sin t & 0 \le t < 1 \\ 2 + 2\sin(t-1) - \sin t & t \geq 1 \end{cases}$$

This is the unique solution to the IVP $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$.

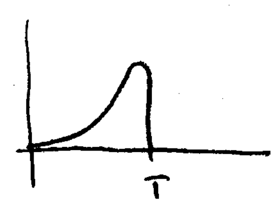
Periodic forcing terms

- Suppose $f(t)$ is periodic. We want to compute $F(s) = \mathcal{L}\{f(t)\}(s)$.

e.g.,

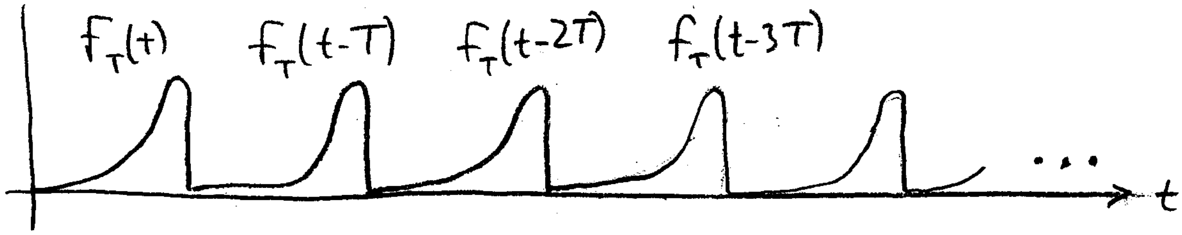


Approach: Consider the "window" $f_T(t)$, defined on $0 \le t < T$, and extended to be periodic.



Then, the function $f(t) = \begin{cases} f_T(t) & 0 \le t < T \\ f_T(t-kT) & kT \le t < (k+1)T \end{cases}$

is periodic.



Big idea: $F(s) = \mathcal{L}\{f(t)\}(s) = \dfrac{F_T(s)}{1-e^{-Ts}} = \dfrac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-Ts}}$

Proof: (key point: If $|x|<1$, then $1+x+x^2+\dots = \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$)

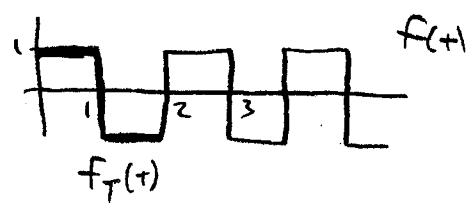
$$f(t) = \sum_{k=1}^{\infty} f_T(t-kT) = \sum_{k=1}^{\infty} f_T(t-kT)\,H(t-kT)$$

$$F(s) = \sum_{k=1}^{\infty} \mathcal{L}\{f_T(t-kT)\,H(t-kT)\}(s)$$

$$= \sum_{k=1}^{\infty} e^{-kTs} F_T(s) = F_T(s)\sum_{k=1}^{\infty} e^{-kTs}$$

$$= F_T(s)\sum_{k=1}^{\infty} \left(e^{-sT}\right)^k = \boxed{F_T(s)\,\frac{1}{1-e^{-sT}}}$$

Example: Solve the IVP $y''+y=f(t)$, $y(0)=0$, $y'(0)=0$, where $f(t)$ is a square wave of period $T=2$.



First, compute $F(s)$:

$$f_T(t) = H_{01}(t) - H_{12}(t) = [H(t)-H(t-1)] - [H(t-1)-H(t-2)]$$

$$= H(t) - 2H(t-1) + H(t-2)$$

$$F_T(s) = \mathcal{L}(H(t)) - 2\mathcal{L}(H(t-1)) + \mathcal{L}(H(t-2))$$

$$= \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s} = \boxed{\frac{(1-e^{-s})^2}{s}}$$

Now, $F(s) = F_T(s) \frac{1}{1-e^{2s}} = \frac{F_T(s)}{(1-e^{-s})(1+e^{-s})}$

$$= \frac{(1-e^{-s})(1-e^{-s})}{s(1-e^{-s})(1+e^{-s})} = \boxed{\frac{(1-e^{-s})}{s(1+e^{-s})}}$$

Back to the IVP: $\mathcal{L}(y'') + \mathcal{L}(y) = F(s)$

$$\left[s^2 Y - s\,y(0) - y'(0)\right] + Y = \frac{1-e^{-s}}{s(1+e^{-s})}$$

$$(s^2+1)Y = \frac{1-e^{-s}}{s(1+e^{-s})} \implies \boxed{Y(s) = \frac{1-e^{-s}}{s(s^2+1)(1+e^{-s})}}$$

Simplify this: $Y(s) = \underbrace{\frac{1}{s(s^2+1)}}_{=\frac{1}{s}-\frac{s}{s^2+1}} \cdot \underbrace{\frac{1-e^{-s}}{1+e^{-s}}}_{\text{need to further simplify this}}$

$$\frac{1-e^{-s}}{1+e^{-s}} = \frac{-(1+e^{-s})}{1+e^{-s}} + \frac{2}{1+e^{-s}} = 2\left(\frac{1}{1-(-e^{-s})}\right) = 2\sum_{n=0}^{\infty}(-1)^n e^{-ns}$$

Note: $\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

Thus, $Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)\left(-1 + 2\sum_{n=0}^{\infty}(-1)^n e^{-ns}\right)$

$$= \underbrace{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)}_{\text{Call this } G(s).}\left(+1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}\right)$$

Note that $\boxed{g(t) = 1 - \cos t}$

Apply $\mathcal{L}\{g(t-n)H(t-n)\}(s) = e^{-ns}G(s)$

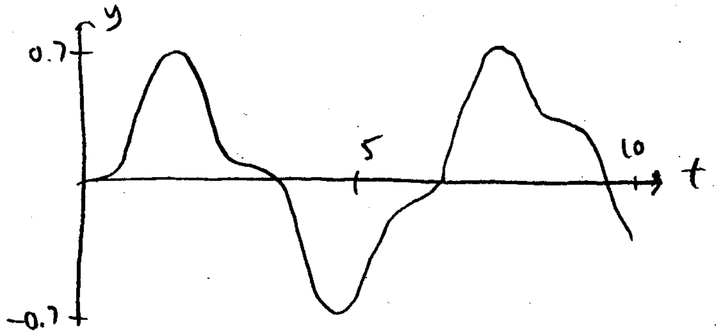
So, $Y(s) = G(s) + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns} G(s)$

and $y(t) = g(t) + 2\sum_{n=1}^{\infty}(-1)^n g(t-n)H(t-n)$

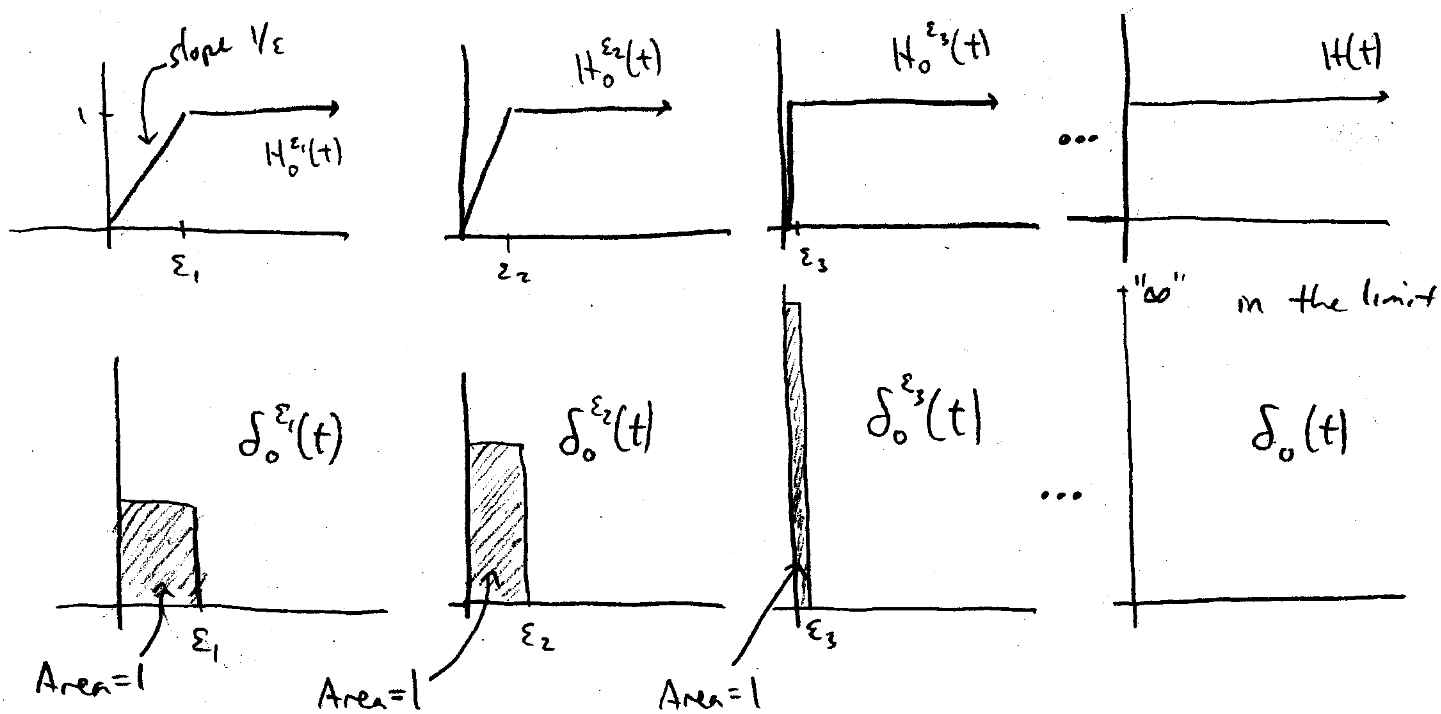
6

$$y(t) = (1-\cos t)\, H(t) + 2 \sum_{n=1}^{\infty} (-1)\left[1-\cos(t-n)\right] H(t-n)$$

This is a superposition of infinitely many waves



<u>Question</u>: What is the "derivative" of the Heavyside function? Technically, it's not defined, but what "should" it be?



<u>Def</u>: The <u>delta function</u> is $\delta_p(t) := \lim_{\varepsilon \to 0} \delta_p^{\varepsilon}(t)$

Technically, it's not really a function (Engineers like to cheat!)

<u>But it's useful</u>:

- $\int_0^\infty \delta_p(t)\, f(t)\, dt = f(p)$
- $\mathcal{L}(\delta_p)(s) = e^{-sp}$
- $\mathcal{L}(\delta_0)(s) = 1$ (so now you can take the inverse Laplace transform of a constant, or exponential).

* This models a <u>unit impulse force</u> (finite force over an infintesimal time interval).

e.g., Exerting a force by hitting something with a hammer.

<u>Example</u>: Solve the IVP $y'' + 2y' + 2y = \delta_0(t)$, $y(0) = 0$, $y'(0) = 0$.

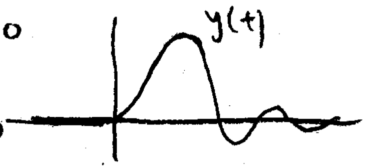
First, take the Laplace transform: $\mathcal{L}(y'' + 2y' + 2y) = \mathcal{L}(\delta_0(t))$

$$(s^2 + 2s + 2)Y = 1$$

$$\Rightarrow Y(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1} \Rightarrow y(t) = e^{-t} \sin t.$$

<u>Note:</u> $y(0) = 0$ but $y'(0) = 1$, (This doesn't match our init. conds!)

To "fix" this, let's use $H(t)\,y(t) = \begin{cases} 0 & t \le 0 \\ e^{-t}\sin t & t > 0 \end{cases}$



This isn't even differentiable at $t = 0$, so technically, the derivative isn't defined. But $\lim_{t \to 0^-} \frac{d}{dt}[H(t)\,y(t)] = 0$, and that's "good enough."