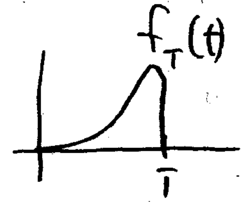
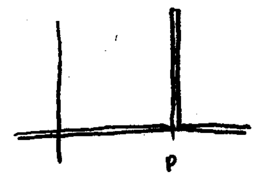
MTHSC 208, WEEK 13

Week 12 summary



- If $f(t)$ is periodic with period T and "window"

  then $\mathcal{L}\{f(t)\}(s) = \dfrac{F_T(t)}{1-e^{-sT}} = F_T(s)\left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right]$



- Delta function $\delta_p(t) = \begin{cases} \text{"}\infty\text{"} & t = p \\ 0 & t \neq p \end{cases} = \lim\limits_{\varepsilon \to 0} \delta_p^{\varepsilon}(t)$

  (not really a function, though).

  * Allows us to take $\mathcal{L}^{-1}$ of a constant

    $\mathcal{L}(\delta_p(t)) = e^{-sp}$, $\quad \mathcal{L}(\delta_0(t)) = 1$

  * $\int_0^\infty \delta_p(t)\, f(t)\, dt = f(p) \qquad \int_0^\infty \delta_p(t)\, dt = 1$

  * Models a <u>unit impulse force</u> (Finite force applied over an instantaneous time period).

<u>Last 3 ½ weeks of class</u>: Partial differential equations – equations involving a multivariate function & its partial derivatives.

e.g., $\dfrac{\partial}{\partial t} u(x,t) = \dfrac{\partial^2}{\partial x^2} u(x,t)$, or just $u_t = u_{xx}$.

To solve these equations, we need to use <u>Fourier series</u>, which we'll study next.

<u>Big idea</u>: Every periodic function (<u>think</u>: arbitrary sound wave) can be decomposed into sine & cosine waves.

We'll learn how to do this.

<u>Motivation</u>: $\mathbb{R}^n$ is a set of <u>vectors</u>.

We can add & subtract vectors, and we know how to "measure" their lengths: $\boxed{\|\vec{v}\| := \sqrt{\vec{v}\cdot\vec{v}}}$

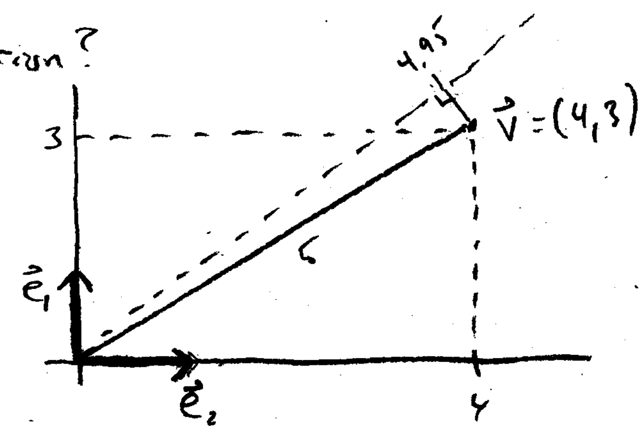
e.g., $\|(4,3)\| = \sqrt{4^2+3^2} = 5$

We can also <u>project</u> a vector onto a <u>unit vector</u> using the <u>dot product</u>.

<u>Example</u>: Let $\vec{v}=(4,3)$, and let $\vec{e}_1=(1,0)$, $\vec{e}_2=(0,1)$ "unit basis vectors"

<u>Q</u>: How long is $\vec{v}$ in the x-direction?

<u>A</u>: $\vec{v}\cdot\vec{e}_1 = (4,3)\cdot(1,0) = 4$



<u>Q</u>: How long is $\vec{v}$ in the y-direction?

<u>A</u>: $\vec{v}\cdot\vec{e}_2 = (4,3)\cdot(0,1) = 3$

<u>Q</u>: How long is $\vec{v}$ in the "northeast," or $(1,1)$-direction?

<u>A</u>: $\vec{v}\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = (4,3)\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = \frac{7\sqrt{2}}{2} \approx 4.95$,

The unit basis vectors $\{\vec{e}_1,\dots,\vec{e}_n\}$ of $\mathbb{R}^n$ have some nice properties:

(i) $\|\vec{e}_i\| = \sqrt{\vec{e}_i\cdot\vec{e}_i} = 1$ ($\vec{e}_i$ has length 1)

(ii) If $i\neq j$, then $\vec{e}_i\cdot\vec{e}_j = 0$. ($\vec{e}_i$ & $\vec{e}_j$ are <u>orthogonal</u> (perpendicular)).

Together, we can summarize this by: $\boxed{\vec{e}_i\cdot\vec{e}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}}$

Def: A set of vectors is <u>orthonormal</u> if they satisfy conditions (i) & (ii) above.

* $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is an <u>orthonormal basis</u> of $\mathbb{R}^n$.

Because of this, we can <u>decompose</u> any vector into components, by projecting onto the basis vectors.

e.g.; $\vec{v} = (5, 4, 3) = 5\vec{e}_1 + 4\vec{e}_2 + 3\vec{e}_3 = (\vec{v}\cdot\vec{e}_1)\vec{e}_1 + (\vec{v}\cdot\vec{e}_2)\vec{e}_2 + (\vec{v}\cdot\vec{e}_3)\vec{e}_3$.

This is the technique that we'll use to decompose a periodic function into sine & cosine waves!

* Let $Per_{2\pi}$ be the set of 2π-periodic piecewise continuous functions.

e.g.,  etc.

We can think of these functions as <u>vectors</u>.

We can add & subtract these "vectors" & multiply by scalars.

We need to define a "dot product" (called an <u>inner product</u>) so we can measure their lengths.

<u>Define</u> $\boxed{\langle f(x), g(x)\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx}$

<u>Note</u>: $\langle f, g\rangle$ is just a preferred notation for "$f \cdot g$".

This defines "length": $\|f(x)\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\, dx$

* <u>Key Fact</u>: The set $\mathcal{B}_{2\pi} := \left\{ \begin{matrix} \frac{1}{\sqrt{2}}, & \cos x, \ \cos 2x, \ \cos 3x, \ldots \\ & \sin x, \ \sin 2x, \ \sin 3x, \ldots \end{matrix} \right\}$

is an <u>orthonormal basis</u> for $Per_{2\pi}$, given our definition of length!

i.e., $\langle \cos nx, \cos mx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

$\langle \sin nx, \sin mx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

$\langle \cos nx, \sin mx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0.$

Now, we automatically know how to decompose a periodic function into sines & cosines – just "project" onto the basis vectors

Let $f(x)$ be a piecewise continuous 2π-periodic function.

Write $\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}$

a_n = "length of $f(x)$ in the $(\cos nx)$-direction"

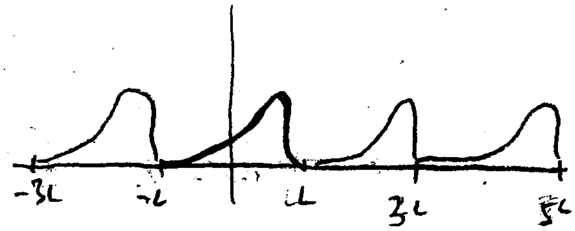
b_n = "length of $f(x)$ in the $(\sin nx)$-direction"

$$\boxed{\begin{aligned} a_n &= \langle f(x), \cos nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n &= \langle f(x), \sin nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{aligned}}$$

Remark: This formula works for a_0 too: $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$.

Note: This easily generalizes to functions of period $2L$ (i.e, not just 2π):

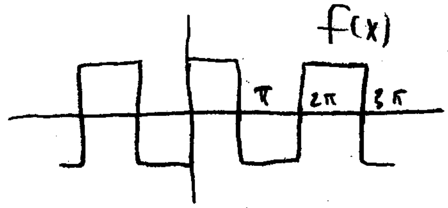
$$\boxed{\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \\ a_n &= \frac{2}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n &= \frac{2}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \end{aligned}}$$



[Show demo: www.falstad.com/fourier]

However, the math is messier for $L \neq \pi$, so we'll just stick with 2π-periodic functions in this class.

Example 1: Square wave:



Find the Fourier series of $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{0} -1\, dx + \frac{1}{\pi}\int_{0}^{\pi} 1\, dx = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{0} -1 \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} 1 \cos nx\, dx$$

$$= -\frac{1}{n\pi} \sin nx \Big|_{-\pi}^{0} + \frac{1}{n\pi} \sin nx \Big|_{0}^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{0} -1 \sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} 1 \sin nx\, dx$$

$$= \frac{1}{n\pi} \cos nx \Big|_{-\pi}^{0} - \frac{1}{n\pi} \cos nx \Big|_{0}^{\pi} = \frac{1}{n\pi}(1 - \cos n\pi) - \frac{1}{n\pi}(\cos n\pi - 1)$$

$$= \boxed{\frac{2}{n\pi}(1 - \cos n\pi)}$$

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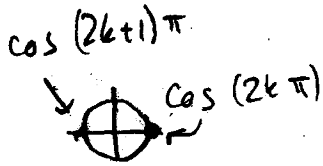
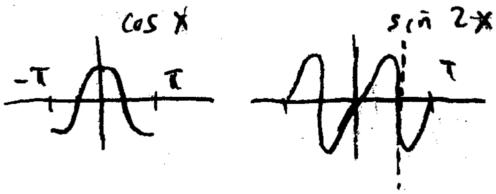


Note: $\cos n\pi = (-1)^n$

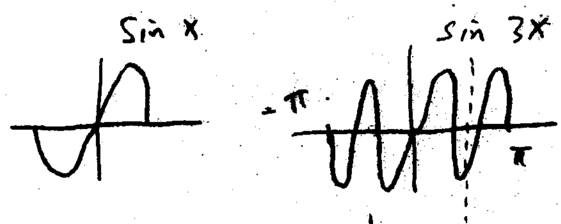
Therefore, $b_n = \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$

i.e., $\boxed{f(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + \dots}$

Note: All cosine terms, and "even-index" sine terms are zero. (Why?)

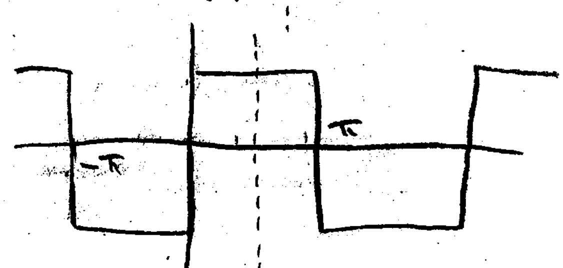


vs.

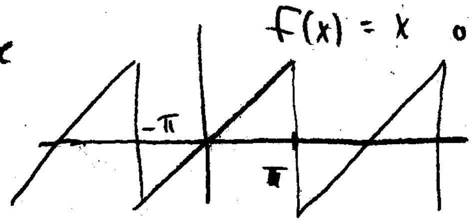


look at the "symmetries" of $f(x)$

This "looks" like a sine wave.



Example 2: Sawtooth wave $\quad f(x) = x$ on $[-\pi, \pi]$



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0 \quad (\text{By symmetry; area under the curve})$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \; dx \qquad \text{let } u = x \quad v = \frac{1}{n}\sin nx, \quad du = dx$$

$$= \frac{1}{\pi}\left[\cancel{\frac{1}{n} x \sin nx \Big|_{-\pi}^{\pi}}^{\,0} - \frac{1}{n}\int_{-\pi}^{\pi} \sin nx\,dx\right]$$

$$= \frac{-1}{n\pi}\int_{-\pi}^{\pi} \sin nx = \frac{1}{n^2\pi}\cos nx\Big|_{-\pi}^{\pi} = \frac{1}{n^2\pi}\left[\cos(\pi x) - \cos(-\pi x)\right]$$

$$= \frac{1}{n^2\pi}\left[\cos \pi x - \cos \pi x\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \qquad \begin{matrix} \text{let } u = x & v = -\frac{1}{n}\cos nx \\ du = dx & dv = \sin nx \, dx \end{matrix}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} x \cos nx \Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx \, dx\right]$$

$$= \frac{1}{\pi n}\left[\left(-\frac{\pi}{n}\cos n\pi\right) - \left(\frac{\pi}{n}\cos n\pi\right) + \frac{1}{n^2}\cancelto{0}{\sin nx \Big|_{-\pi}^{\pi}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos(n\pi)\right] = -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} = \begin{cases} -2/n & n \text{ even} \\ 2/n & n \text{ odd} \end{cases}$$

Thus, $f(x) = 2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \frac{2}{5}\sin 5x \pm \cdots$

$$= 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \frac{2}{5}\sin 5x \pm \cdots$$

Think: How does this relate to music, sound waves, & harmonics?

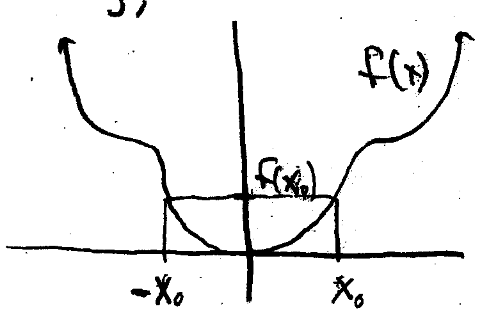
Exploiting Symmetry

Why are many of the a_n's & b_n's zero?

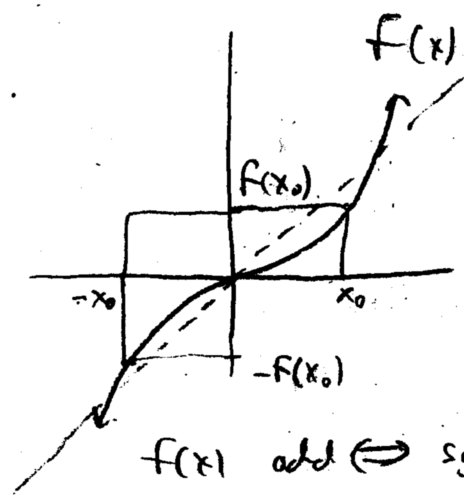
Def: • $f(x)$ is an even function if $f(x) = f(-x)$

• $f(x)$ is an odd function if $f(x) = -f(-x)$

Graphically,



$f(x)$ even $\Leftrightarrow$ symmetric about the y-axis



$f(x)$ odd $\Leftrightarrow$ symmetric about the origin.

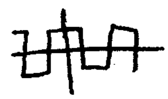
<u>Why we care:</u>

- If $f(x)$ is even, then $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$
- If $f(x)$ is odd, then $\int_{-L}^{L} f(x)\,dx = 0$

Look at the area under the curve to see why!

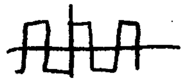
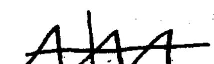
<u>Basic facts:</u>
- If f & g are <u>even</u>, then $f(x)\,g(x)$ is <u>even</u>
- If f & g are odd, then $f(x)\,g(x)$ is <u>even</u>
- If f is even, g is odd, then $f(x)\,g(x)$ is <u>odd</u>.

<u>Examples:</u>

- <u>Even functions:</u> 8, x^2, $3x^6 + x^2 - 5$, $|x|$, 

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +/- \cdots = \frac{e^{ix} + e^{-ix}}{2}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} +/- \cdots = \frac{e^{x} + e^{-x}}{2}$$

- <u>odd functions:</u> $2x$, $8x^3 - 5x$,  

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +/- \cdots = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \frac{e^{x} - e^{-x}}{2}$$

- <u>Neither:</u> $x^2 - 3x + 2$, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

Off-hand remark:

* $\cos x = \cosh ix$
* $\sin x = \frac{1}{i}\sinh ix$
* $e^x = \cosh x + \sinh x = \cos x + i \sin x$

Remark:

- If $f(x)$ is even, then $f(x)\cos nx$ is even $\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$

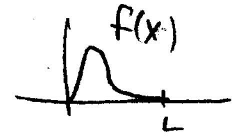
  and $f(x)\sin nx$ is odd $\Rightarrow b_n = 0$ (all n)

- If $f(x)$ is odd, then $f(x)\cos nx$ is odd $\Rightarrow a_n = 0$ (all n)

  and $f(x)\sin nx$ is even $\Rightarrow b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$.

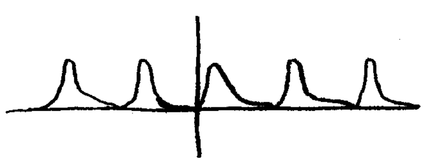
Fourier sine & cosine series

Idea: Consider some function defined on $[0,L]$

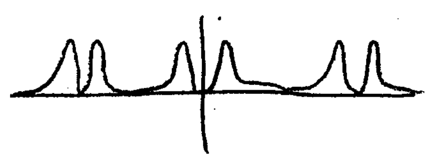


Find "the Fourier series of $f(x)$."

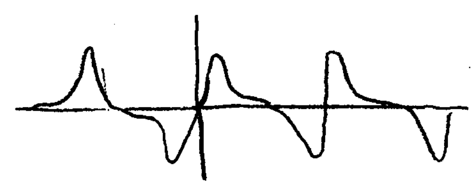
First, we need to make $f(x)$ periodic.



A naive extension



The even extension



The odd extension

- The Fourier cosine series is the Fourier series of the even extension of $f(x)$.

$$\begin{cases} a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = 0 \end{cases}$$

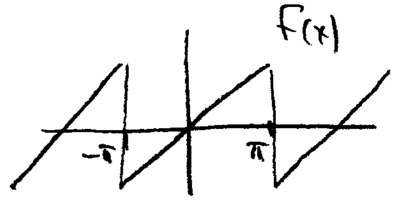
- The Fourier sine series is the Fourier series of the odd extension of $f(x)$

$$\begin{cases} a_n = 0 \\ b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases}$$

Example 3: Let $f(x) = x$ on $[0, \pi]$.

Compute the Fourier sine & cosine series of $f(x)$.

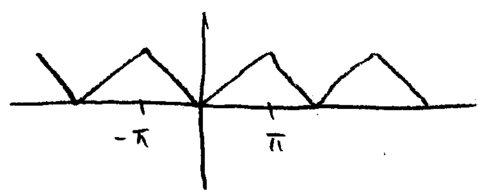
Fourier sine series: Odd extension:



This was Example 2 on p. 6-7.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \, , \qquad b_n = \begin{cases} -2/n\pi & n \text{ even} \\ 2/n\pi & n \text{ odd} \end{cases}$$

Fourier cosine series: Even extension



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \frac{x^2}{\pi}\Big|_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi}\left[\frac{x}{n} \underbrace{\cancel{\sin nx}\Big|_0^{\pi}}_{0} - \int_0^{\pi} \frac{1}{n} \sin nx \, dx \right]$$

Let $u = x$ $\quad v = \frac{1}{n}\sin nx$

$du = dx$ $\quad dv = \cos nx \, dx$

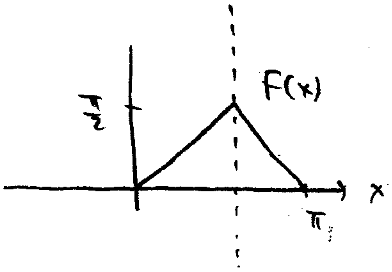
$$= \frac{2}{\pi n^2} \cos nx \Big|_0^{\pi} = \frac{2}{n^2 \pi}\left[\cos n\pi - 1\right]$$

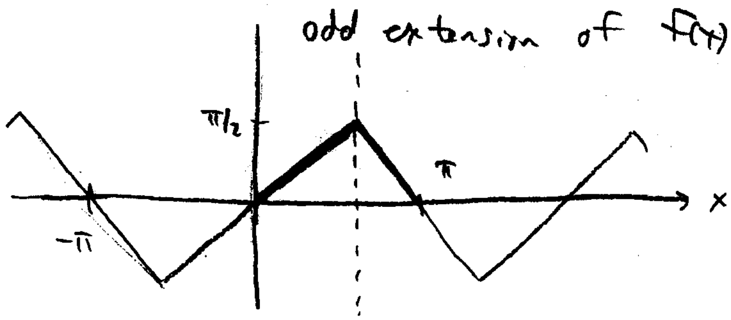
$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{\pi n^2} & n \text{ odd} \end{cases}$$

Thus, $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25}\cos 5x - \frac{4}{49\pi}\cos 7t - \cdots$.

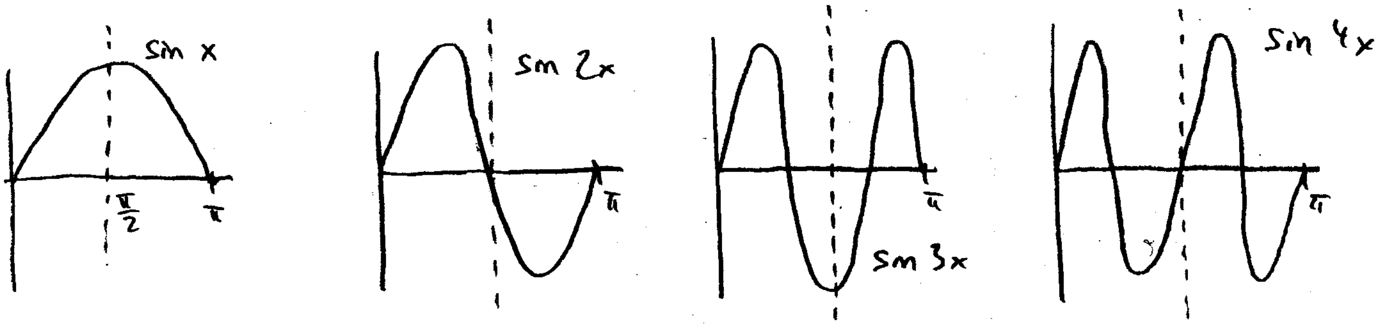
Example 4: Let $f(x) = \begin{cases} x & 0 \le x < \pi/2 \\ \pi - x & \pi/2 \le x < \pi \end{cases}$

Compute the Fourier sine series of $f(x)$.





Observe the symmetry about the line $x = \pi/2$.



* $\sin nx$ has "even symmetry" about $x = \pi/2$" if n is odd
* $\sin nx$ has 'odd symmetry" about $x = \pi/2$' if n is even.

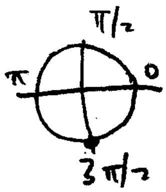
Conclusion: $b_n = 0$ for all even n, and if n is odd, then

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{4}{\pi}\int_0^{\pi/2} f(x)\sin nx\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} x\sin nx\,dx = \frac{4}{\pi}\left[\frac{x}{n}\cos nx\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{n}\cos nx\,dx\right]$$

$$= \frac{4}{\pi}\left[\frac{\pi}{2n}\overbrace{\cancel{\cos\left(\frac{n\pi}{2}\right)}}^{0} - 0 + \frac{1}{n^2}\sin nx\Big|_0^{\pi/2}\right] \quad \text{(since } n \text{ is odd).}$$

$$= \frac{4}{\pi}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$



Note: $\sin\frac{n\pi}{2} = \begin{cases} 0 & n = 4k \\ 1 & n = 4k+1 \\ 0 & n = 4k+2 \\ -1 & n = 4k+3 \end{cases}$

Thus, $b_n = \begin{cases} 0 & n = 4k \\ 4/n^2\pi & n = 4k+1 \\ 0 & n = 4k+2 \\ -4/n^2\pi & n = 4k+3 \end{cases}$

So, $f(x) = \frac{4}{\pi}\sin x - \frac{4}{9\pi}\sin 3x + \frac{4}{25\pi}\sin 5x - \frac{4}{49\pi}\sin 7x +/- \dots$