MTHSC 208, WEEK 14

Week 13 summary:

- A dot product, or "inner product" allows us to project vectors onto unit vectors.

- For Per_{2L}, define $\langle f(x), g(x) \rangle = \frac{1}{L}\int_{-L}^{L} f(x)\, g(x)\, dx$

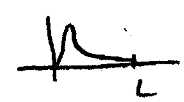
  * If $f(x)$ is 2π-periodic, then $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

    $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$

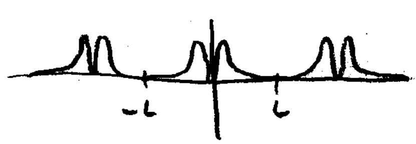
- **Even functions**
  * $f(x) = f(-x)$
  * Symmetric about y-axis
  * Fourier series contains only sines
  * $\int_{-L}^{L} f(x)\, dx = 2\int_0^L f(x)\, dx$

  Odd functions
  * $f(x) = -f(-x)$
  * Symmetric about origin
  * Fourier series contains only cosines
  * $\int_{-L}^{L} f(x)\, dx = 0.$

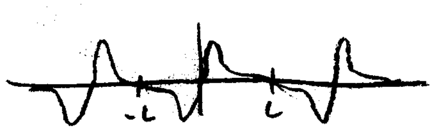
- Fourier cosine & sine series.

  Start with a function $f(x)$ on $[0, L]$, e.g., 

  * Fourier cosine series is the Fourier series of the even extension

    e.g.,  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L}\int_0^{\pi} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

  * Fourier sine series is the Fourier series of the odd extension.

    e.g.,  $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad b_n = \frac{2}{L}\int_0^{\pi} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

Complex form of the Fourier series

Fact 1: $\mathcal{B}_1 = \left\{ \begin{array}{l} \frac{1}{\sqrt{2}},\ \cos x,\ \cos 2x,\ \cos 3x, \ldots \\ \phantom{\frac{1}{\sqrt{2}},\ } \sin x,\ \sin 2x,\ \sin 3x, \ldots \end{array} \right\}$ is a basis for $Per_{2\pi}$.

and is orthonormal if $\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

Fact 2: $\mathcal{B}_2 = \left\{ \begin{array}{l} 1,\ e^{-ix},\ e^{-2ix},\ e^{-3ix}, \ldots \\ e^{ix},\ e^{2ix},\ e^{3ix}, \ldots \end{array} \right\}$ is also a basis for $Per_{2\pi}$

and is orthonormal if $\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$.

Therefore, if $f(x)$ is 2π-periodic, we can write it as

$$\boxed{\begin{array}{l} f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{-inx} + c_{-n} e^{-inx} \right) \\[2ex] c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \qquad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \end{array}}$$

This is the complex form of the Fourier series of $f(x)$.

Recall: $\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$, $\quad \sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$.

$$e^{inx} = \cos nx + i \sin nx, \qquad e^{-inx} = \cos nx - i \sin nx$$

Therefore, $\boxed{c_n = \frac{a_n - i b_n}{2}, \qquad c_{-n} = \frac{a_n + i b_n}{2}}$

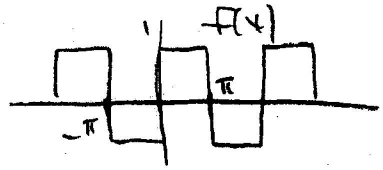
and $\boxed{a_n = c_n + c_{-n} \qquad b_n = i\left(c_n - c_{-n}\right)}$

Note: c_0 is the const. term in the complex form of $f(x)$.

$a_0 = 2c_0 \Rightarrow \frac{a_0}{2}$ is the const. term in the real form.

Remark: The constant term c_0 (or $\frac{a_0}{2}$) is the average value of $f(x)$. (Why?)

Example 1: Compute the complex Fourier series of



$$\left(\text{i.e., } f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}\right)$$

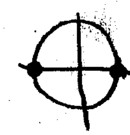
$c_0 = 0$ (average value of $f(x)$).

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} -e^{inx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{in} e^{-inx}\right]_0^{\pi} + \frac{1}{2\pi}\left[\frac{-1}{in} e^{-inx}\right]_0^{\pi}$$

$$= \frac{1}{2\pi i n}\left(1 - e^{in\pi} - e^{-inx} + 1\right) \qquad \underline{\text{Note:}}\ e^{-inx} = e^{inx} = (-1)^n = (-1)^{-n}$$

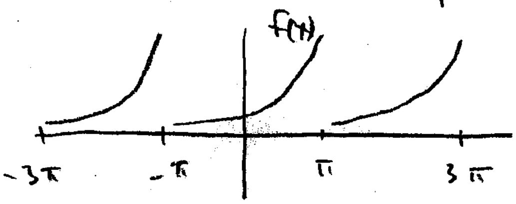
$$= \boxed{\frac{1}{\pi i n}\left(1-(-1)^n\right)} = \begin{cases} \frac{2}{\pi i n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$



$\mathbb{C}$

Thus, $$\boxed{f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{\pi i n}\left(1-(-1)^n\right) e^{inx} = \sum_{n=1}^{\infty} \frac{1}{\pi i n}\left(1-(-1)^n\right)\left(e^{inx} - e^{inx}\right)}$$

Example 2: Compute the complex Fourier series of the 2π-periodic extension of e^x.



$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{2\pi} e^x\Big|_{-\pi}^{\pi} = \boxed{\frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right)}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx = \frac{1}{2\pi(1-in)} e^{(1-in)x}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-in)}\left[e^{(1-in)\pi} - e^{-(1-in)\pi}\right] = \frac{e^{in\pi}}{2\pi(1-in)}\left[e^{\pi} - e^{-\pi}\right].$$

$$= \frac{(-1)^n}{2\pi(1-in)}\left[e^{\pi} - e^{-\pi}\right]$$

Note: $\frac{1}{1-in} = \frac{1}{1-in}\,\frac{1+in}{1+in} = \frac{1+in}{1+n^2} \Rightarrow \boxed{c_n = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{2\pi(1+n^2)}(1+in)}$

Now, derive the real Fourier coefficients:

$$a_n = c_n + c_{-n} = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}$$

$$b_n = i(c_n - c_{-n}) = \frac{(-1)^n n(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}$$

Parseval's identity: If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Proof: $$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\underbrace{\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx\right)}_{f(x)}dx$$

$$= \frac{a_0}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\,dx$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\Big(a_n\cdot \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}_{a_n} + b_n \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}_{b_n}\Big)$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

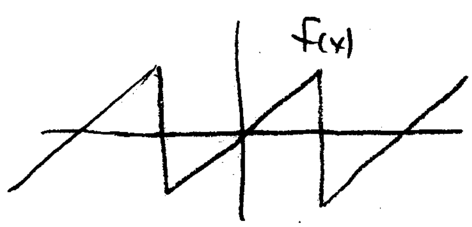
$\square$

Neat application: Compute $\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

Let $f(x) = x$ on $[-\pi, \pi]$.

$a_n = 0$ (since $f(x)$ is odd)

$b_n = \frac{2}{n}(-1)^n$ (last week)



$\Rightarrow b_n^2 = \frac{4}{n^2}$

Apply Parseval's identity: $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}$ (LHS)

RHS: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}$

Equate LHS & RHS: $\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3} \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$

Partial differential equations

Let $u(x,t)$ be a 2-variable function. A partial differential equation (PDE) is an equation involving u, x, t, and the partial derivatives of u.

Example: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ (or just $u_t = u_{xx}$).

ODEs have a unifying theory of existence & uniqueness of solutions

PDEs have no such theory.

PDEs arise from physical phenomena & modeling.

Heat equation: $\rho(x)\,\sigma(x)\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\,\frac{\partial u}{\partial x}\right)$, where

$u(x,t)$ = temperature of a bar at position x & time t

$\rho(x)$ = density of bar at position x

$\sigma(x)$ = specific heat at position x

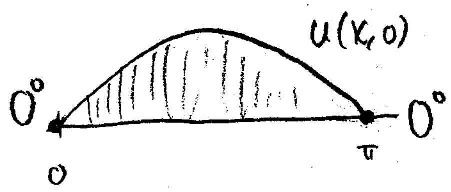
$\kappa(x)$ = thermal conductivity of the bar at position x

* We'll assume the bar is homogeneous (i.e., ρ, σ, κ constants)

In this case, the heat equation becomes

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}} \qquad \text{where } c^2 = \frac{K}{\rho\sigma}$$

Example 1: Let $u(x,t)$ = temp. of a bar of length π, insulated along the sides, whose ends are kept at zero temp. (Boundary conditions), and $u(x,0) = x(\pi - x)$ (Initial conditions).



Thus, we have the following initial value problem:

$$u_t = c^2 u_{xx}, \quad u(0,t) = 0, \quad u(\pi, t) = 0, \quad u(x,0) = x(\pi - x).$$

Note: This is homogeneous and linear, i.e, if u_1 & u_2 are solutions, then so is $c_1 u_1 + c_2 u_2$ (superposition).

Let's solve this!

Step 1: Find the general solution to $u_t = c^2 u_{xx}$.

* Assume $u(x,t) = f(x)\, g(t)$; "Separation of variables"

$u_t = f(x)\, g'(t)$, $\quad u_{xx} = f''(x)\, g(t)$

Now, plug back in & solve for f & g.

$$u_t = c^2 u_{xx} \Longrightarrow f(x)\, g'(t) = c^2 f''(x)\, g(t)$$

$$\Rightarrow \frac{g'(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda$$

Therefore this must be a <u>constant</u>!

Doesn't depend on x — Doesn't depend on t

Now, we have 2 ODE's: $\dfrac{g'(t)}{c^2 g(t)} = \lambda, \quad \dfrac{f''(x)}{f(x)} = \lambda.$

<u>Solve for g</u>: $g' = c^2 \lambda g \Rightarrow \boxed{g(t) = Ae^{c^2\lambda t}}$ ✓

Suppose $g(t) \neq 0$. Boundary conditions: $u(0,t) = u(\pi,t) = 0$ imply that $f(0)g(t) = f(\pi)g(t) = 0 \Rightarrow f(0) = 0$ & $f(\pi) = 0$ (why?)

<u>Solve for f</u>: $f'' = \lambda f, \quad f(0) = 0, \quad f(\pi) = 0.$

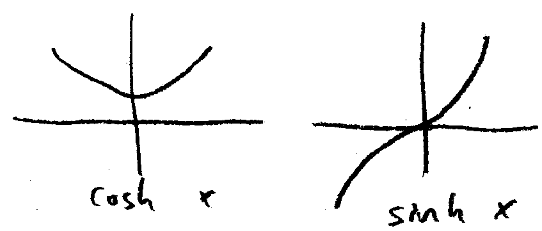
<u>Case 1</u>: $\boxed{\lambda = 0}$ $\quad f'' = 0 \Rightarrow f(x) = ax + b, \quad f(0) = 0 \Rightarrow a = 0$

$f(\pi) = 0 \Rightarrow b = 0 \Rightarrow f(x) = 0.$

<u>Case 2</u>: $\boxed{\lambda > 0}$ $\quad f(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

<u>OR</u> $f(x) = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$ [See HW #19]

<u>Recall</u>: $\cosh 0 = 1, \quad \sinh 0 = 0$



$f(0) = A = 0 \Rightarrow f(x) = B\sinh(\sqrt{\lambda}x)$

$f(\pi) = B\sinh(\sqrt{\lambda}\pi) = 0 \Rightarrow B = 0$

$\Rightarrow f(x) = 0.$

<u>Case 3</u>: $\boxed{\lambda < 0}$ $\quad$ Let $\omega = \sqrt{-\lambda}$, so $f'' = -\omega^2 f$

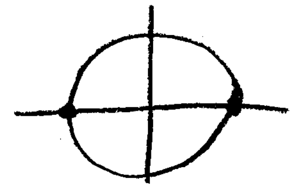
Simple harmonic motion: $f(x) = a\cos\omega x + b\sin\omega x$

$f(0) = 0, \quad f(\pi) = 0$

$f(0) = a = 0 \Rightarrow f(x) = b\sin\omega x$

$f(\pi) = b \sin w\pi = 0 \implies w\pi = n\pi$

$\implies \boxed{w = n}$



$(\sin w\pi = 0$ iff $w\pi = n\pi)$

Therefore, $\boxed{f(x) = b \sin nx}$, for any integer n.

In summary, for any fixed choice of $\lambda = -n^2$, we have a solution $u_n(x,t) = f_n(x)\, g_n(t)$, where $g_n(t) = A_n e^{-c^2 n^2 t}$

$f_n(x) = B_n \sin nx$

Thus, $\boxed{u_n(x,t) = b_n e^{-c^2 n^2 t} \sin nx}$ is a solution for any n

(Here, we just "absorb" the constants into one constant, b_n).

By superposition, any linear combination is also a solution.

Thus, the general solution is $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-c^2 n^2 t}} \quad (*)$$

Now, let's solve the initial value problem: $u(x,0) = x(\pi - x)$

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = x(\pi - x)$ on $[0, \pi]$.

To solve for the b_n's, we must write $x(\pi - x)$ as a Fourier sine series.

Recall: $b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx = \frac{4}{\pi n^3}\left(1 - (-1)^n\right)$

(See HW 18)

Thus, $u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx$

$\Rightarrow b_n = \frac{4}{\pi n^3}\left(1-(-1)^n\right).$

Our particular solution to the IVP is

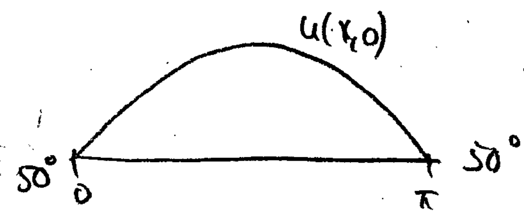
$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx \; e^{-c^2 n^2 t}}$$

Remark: The steady-state solution is $\lim_{t\to\infty} u(x,t) = 0$

(because $e^{-c^2n^2t} \to 0$ as $t \to 0$).

* This makes sense physically, because heat dissipates.

Example 1(b) Consider the same physical situation, but now, say that the boundaries are held fixed at $50°$ (Init. cond. adjusted accordingly).



We now have the following initial value problem:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(\pi,t) = \underline{\underline{50}}, \qquad u(x,0) = x(\pi - x) + \underline{\underline{50}}$$

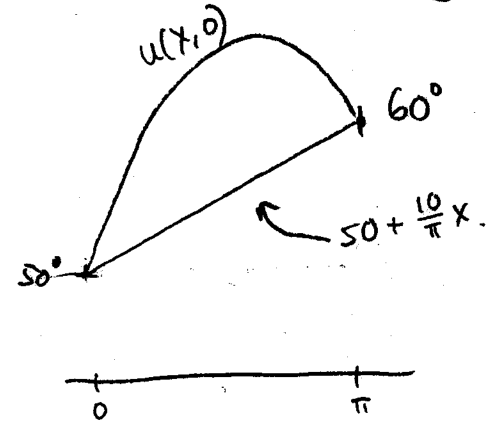
Question: What's the solution? (i.e., how does it differ from the previous example)

Answer: $$\boxed{u(x,t) = 50 + \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx \; e^{-c^2n^2t}}$$

Motivation: This is exactly the same as the previous problem, had we proclaimed $50°$ to be $0°$ (say, in a "new" temperature system).

Remark: $\lim_{t\to\infty} u(x,t) = 50$ is the steady-state sol'n. (this makes sense too!)

<u>Example 1(c):</u> Consider the same physical situation, but now, say that the left-hand boundary is fixed at 50°, and the right-hand boundary is fixed at 60° (& init. cond. adjusted accordingly).



We now have the following initial value problem:

$$u_t = c^2 u_{xx}, \quad u(0,t) = \underline{\underline{50}}, \quad u(\pi,t) = \underline{\underline{60}}, \quad u(x,0) = \underline{\underline{50 + \tfrac{10}{\pi}x}} + x(\pi - x)$$

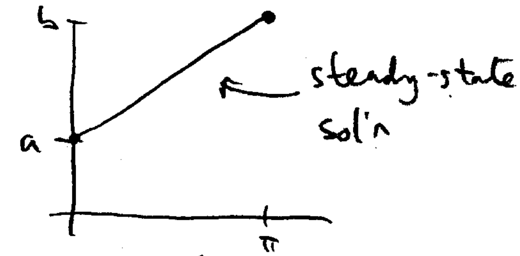
The solution (not surprisingly) is

$$\boxed{u(x,t) = \underbrace{50 + \tfrac{10}{\pi}x}_{\text{steady-state solution}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin nx \, e^{-c^2n^2t}}_{\text{"homogeneous" part}}}$$

i.e, sol'n if the boundary conditions were 0°.

<u>Big idea:</u> To solve $u_t = c^2 u_{xx}$ $u(0,t) = a$, $u(\pi,t) = b$, $u(x,0) = h(x)$, first solve the related problem where $a = b = 0$ then add this to the steady-state solution, which will clearly (why?) be $u_{ss}(x,t) = a + \frac{b-a}{\pi}x$.

i.e, $\boxed{u(x,t) = u_{0,0}(x,t) + u_{ss}(x,t)}$

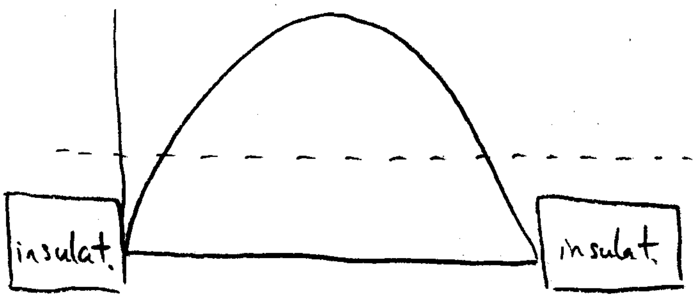


* Convince yourself why this makes sense physically.

$$\lim_{t\to\infty} u_{0,0}(x,t) = 0, \quad \text{so} \quad \lim_{t\to\infty} u(x,t) = u_{ss}(x,t) = a + \frac{b-a}{\pi}x.$$

Example 2: Same situation as Example 1, but with different boundary conditions.

$$u_t = c^2 u_{xx}, \qquad \underbrace{u_x(0,t) = u_x(\pi,t) = 0}_{\text{Represents insulated endpoints, through which no heat can pass.}}, \qquad u(x,0) = x(\pi - x)$$



steady-state sol'n = ave. temperature (as we'll see, this is $\frac{a_0}{2}$!)

To solve this, proceed as before:

Assume $u(x,t) = f(x)g(t)$. $\qquad u_x(0,t) = f'(0)\,g(x) = 0 \Rightarrow f'(0) = 0$.

$\qquad u_x(\pi,t) = f'(\pi)\,g(x) = 0 \Rightarrow f'(\pi) = 0$

Remark: The only difference between this, and Example 1, is

$$f'(0) = f'(\pi) = 0, \quad \text{vs.} \quad f(0) = f(\pi) = 0.$$

g is the same as before: $\quad g(t) = A_n e^{-c^2 \lambda t}$

f has different boundary conditions: $\quad f'' = \lambda f, \quad f'(0) = f'(\pi) = 0$

This has solution $f(x) = a\cos\omega x + b\sin\omega x$.

$$f'(0) = b\omega = 0 \Rightarrow b = 0$$

$$f'(\pi) = a\omega\sin\omega\pi = 0 \Rightarrow \omega\pi = n\pi \Rightarrow \omega = n \text{ (as before)}$$

$$\Rightarrow \boxed{f_n(x) = a_n\cos nx} \quad \text{and} \quad \boxed{g_n(t) = A_n e^{-c^2 n^2 t}} \quad \text{for } n \geq 0.$$

Thus, the general solution becomes:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} f_n(x)\, g_n(t)$$

(Note: when $n=0$, f_n & g_n are constants, not necessarily zero).

$$\Rightarrow \boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \; e^{-c^2 n^2 t}}$$

Now, let's solve the initial value problem: $u(x,0) = x(\pi - x)$.

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \underbrace{x(\pi - x)}$$

we must express this as a Fourier cosine series.

Recall: (HW 18): $a_0 = \frac{\pi^2}{3}$, $a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\, dx = \frac{2}{n^2}\left(1-(-1)^n\right)$.

Thus, $u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}\left(1-(-1)^n\right)\cos nx$

$\Rightarrow \frac{a_0}{2} = \frac{\pi^2}{6}$ and $a_n = \frac{2}{n^2}\left(1-(-1)^n\right)$, so the solution to the IVP is

$$\boxed{u(x,t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}\left(1-(-1)^n\right)\cos nx \; e^{-c^2 n^2 t}}$$