MTHSC 208, Week 15

Week 14 summary:

- Real Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

  Complex Fourier series: $f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{-inx} + c_{-n} e^{inx} = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$

  $e^{inx} = \cos nx + i \sin nx, \quad \cos nx = \frac{e^{inx}+e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx}-e^{-inx}}{2i}$

  Thus, $a_n = c_n + c_{-n}$ $\quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}$

  $b_n = i(c_n - c_{-n})$

- <u>Partial differential equations</u> (PDE's): Equations involving a multivariate function and its partial derivatives.

* <u>Heat equation</u>: $u_t = c^2 u_{xx}$

<u>Boundary conditions</u>:

- <u>Dirichlet</u>: $u(0,t) = a, \quad u(L,t) = b$ (temp. of endpts held fixed).
- <u>Neumann</u>: $u_x(0,t) = u_x(L,t) = 0$ (insulated endpoints).

<u>Initial conditions</u>: $u(x,0) = h(x)$ (Initial heat distribution in the bar)

* <u>Solving PDE's by Separation of Variables</u>

<u>Step</u> 1: Assume $u(x,t) = f(x)\,g(t)$. Plug back in & "separate variables"

<u>Step</u> 2: Set resulting eq'n (e.g., $\frac{f''}{f} = \frac{g'}{c^2 g}$) equal to a constant λ.
Get 2 ODE's: one for $g(t)$ & one for $f(x)$.

Step 3: Solve ODE for $g(t)$ (general sol'n). Solve the ODE for $f(x)$ with boundary conditions (particular sol'n). Find λ.

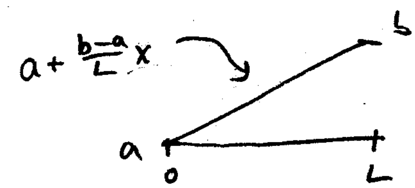
Step 4: The general solution is $u(x,t) = \sum_{n=0}^{\infty} f_n(x)\, g_n(t)$ (by superposition).

Step 5: Plug in $t=0$ & use initial conditions to find the unique solution ($u(x,0)$ will be a Fourier series, you may have to find the Fourier sine / cosine series of $u(x,0) = h(x)$).

* Solving an initial value problem with non-zero boundary conditions:

$$u_t = c^2 u_{xx}, \quad u(0,t) = a, \quad u(L,t) = b, \quad u(x,0) = h(x).$$

- The steady-state solution is $U_{ss}(x) = a + \frac{b-a}{L} x$



- Add this to the solution $U_{0,0}(x,t)$ of the same IVP but where $a=0$, $b=0$.

The final solution is $\boxed{u(x,t) = U_{ss}(x) + U_{0,0}(x,t)}$

This week: The Wave equation: $u_{tt} = c^2 u_{xx}$.

Motivation: Consider the following PDE: $\boxed{\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0}$ $(*)$

Let $F(x)$ be any one-variable function, and set

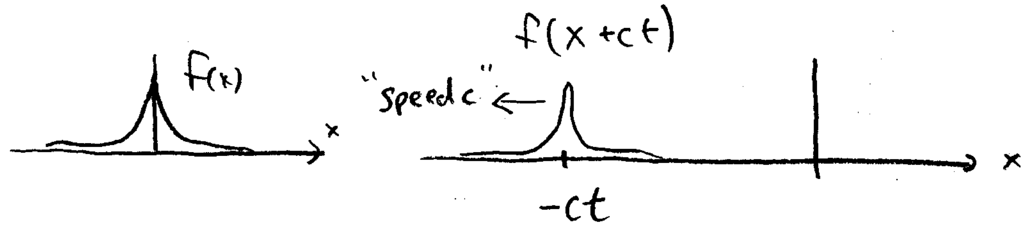
$u(x,t) = f(x+ct)$. Chain rule $\Rightarrow$ $u_x(x,t) = f'(x+ct)$

$u_t(x,t) = c f'(x+ct)$

Note: $u_t - c u_x = c f'(x+ct) - c f'(x+ct) = 0$ ✓

i.e., $f(x+ct)$ is a solution to the PDE in $(*)$.

Picture of this:



As t increases, $u(x,t) = f(x+ct)$ is a <u>traveling wave</u> (to the left, at speed c).

Next, consider the PDE: $\boxed{\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0}$ (**)

Let $g(x)$ be <u>any</u> one-variable function, and set

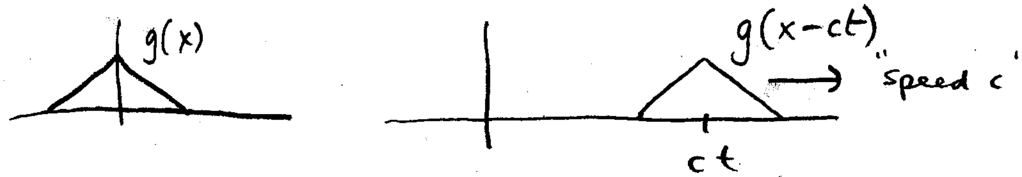
$u(x,t) = g(x-ct)$ Chain rule $\Rightarrow$ $u_x(x,t) = g'(x-ct)$

$u_t(x,t) = -c\,g'(x-ct)$

<u>Note</u>: $u_t + cu_x = -c\,g'(x-ct) + c\,g'(x-ct) = 0$ ✓

i.e., $g(x-ct)$ is a solution to the PDE in (**)

Picture of this:



Now, let $f(x)$ & $g(x)$ be <u>any</u> two one-variable functions. Consider the PDE

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = \boxed{\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0} \quad (***)$$

<u>Check</u>: $u(x,t) = f(x+ct) + g(x-ct)$ is a solution.

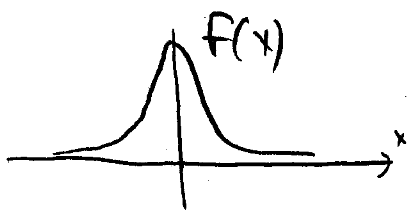
Consider the following <u>initial value problem</u>:

$u_{tt} = c^2 u_{xx}$ (i.e, the PDE in (***))

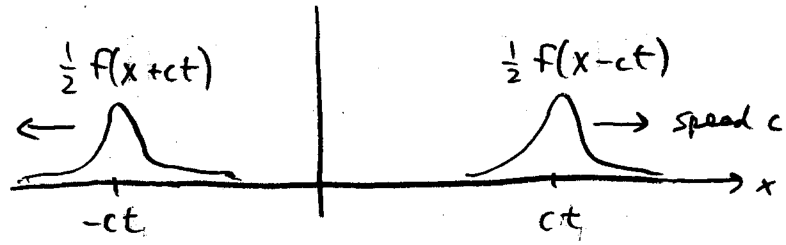
$u(x,0) = f(x)$ "initial <u>displacement</u>, or wave"

$u_t(x,0) = 0$ "initial <u>velocity</u> (vertical ; pointwise)"

<u>Picture of this:</u> Start with a stationary wave in the ocean, at time $t=0$, then "let go." It should disperse, left & right.



initially: $t=0$



At time $t=t_1$ in the future.

The solution to this initial value problem is

$$u(x,t) = \tfrac{1}{2} f(x+ct) + \tfrac{1}{2} f(x-ct)$$

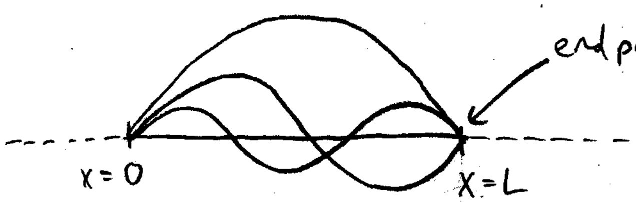
"half the wave (or energy) goes to the left, half goes right."

<u>**Big idea:**</u> $\boxed{\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \text{ is the } \underline{\text{wave}} \text{ equation}}$

- Now, suppose we want to model vibrations (waves) on a finite string/wire of length L.

We need to impose <u>boundary conditions</u>.

endpoints are <u>fixed</u> for all t



Let $u(x,t)$ be the (vertical) displacement at position x & time t.

Fixed endpoints $\Rightarrow u(0,t)=0$ and $u(L,t)=0$.

We must specify the initial wave: $u(x,0)=h_1(x)$

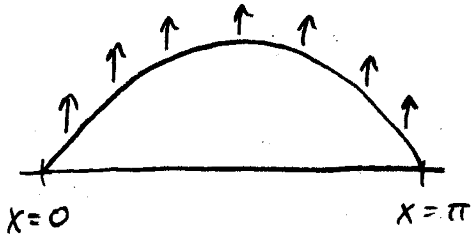
<u>and</u> initial (vertical) velocity @ x: $u_t(x,0)=h_2(x)$.

Together, we get an initial value problem for the <u>wave equation</u>:

$$\boxed{\begin{array}{lll} u_{tt} = c^2 u_{xx} & u(0,t) = 0, & u(L,t) = 0 \\ & u(x,0) = h_1(x), & u_t(x,0) = h_2(x) \end{array}}$$

We solve this using <u>separation of variables</u>, just like the heat equation.

Example: Consider the IVP:



picture at $t=0$

$$u_{tt} = c^2 u_{xx}$$

$$u(0,t) = 0, \quad u(\pi,t) = 0$$

$$u(x,0) = x(\pi - x), \quad u_t(x,0) = 1.$$

<u>Step 1</u>: Assume $u(x,t) = f(x)\,g(t)$ & plug back in:

$$u_{tt} = f g'', \quad u_{xx} = f'' g \Rightarrow f g'' = c^2 f'' g \Rightarrow \frac{f''}{f} = \frac{g''}{c^2 g}$$

<u>Step 2</u>: $\frac{f''}{f} = \frac{g''}{c^2 g} = \lambda \Rightarrow \begin{cases} f'' = \lambda f \\ g'' = c^2 \lambda g \end{cases}$

Moreover, $u(0,t) = f(0)\,g(t) = 0 \Rightarrow f(0) = 0$

$$u(\pi,t) = f(\pi)\,g(t) = 0 \Rightarrow f(\pi) = 0$$

<u>Step 3</u>: Solve the ODEs for $f(x)$ & $g(t)$.

<u>ODE 1</u>: $f'' = \lambda f, \quad f(0) = 0, \quad f(\pi) = 0$

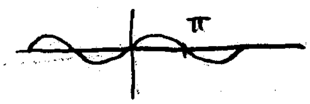
We've done this! $\lambda = -n^2$, $\boxed{f_n(x) = b_n \sin nx}$

<u>ODE 2</u>: $g'' = -c^2 n^2 g \Rightarrow \boxed{g_n(t) = a_n \cos(cnt) + b_n \sin(cnt)}$

Step 4: The general solution (by superposition) is

$$u(x,t) = \sum_{n=0}^{\infty} \underbrace{f_n(x)\, g_n(t)}_{u_n(x,t)} = \boxed{\sum_{n=1}^{\infty} \left(a_n \cos(cnt) + b_n \sin(cnt)\right) \sin nx}$$

Step 5: Plug in $t=0$ & use (both) initial conditions.



(i) $u(x,0) = x(\pi - x)$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx$$

(The Fourier sine series of $x(\pi - x)$).

$$\Rightarrow \boxed{a_n = \frac{4}{\pi n^3}\left(1-(-1)^n\right)}$$

(ii) $u_t(x,0) = 1$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-cn\, a_n \sin(cnt) + cn\, b_n \cos(cnt)\right) \sin nx$$

$$u_t(x,0) = \sum_{n=1}^{\infty} cn\, b_n \sin(nx) = 1 = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right) \sin nx$$

(The Fourier sine series of 1).

$$\Rightarrow cn\, b_n = \frac{2}{n\pi}\left(1-(-1)^n\right) \Rightarrow \boxed{b_n = \frac{2}{cn^2\pi}\left(1-(-1)^n\right)}$$

The solution to this initial value problem for the wave equation is thus

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(cnt) + b_n \sin(cnt)\right) \sin nx, \quad \text{i.e.,}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4}{\pi n^3}\left(1-(-1)^n\right) \cos(cnt) + \frac{2}{\pi c n^2}\left(1-(-1)^n\right) \sin(cnt)\right] \sin nx}$$