MTHSC 208, WEEK 16

Week 15 summary:

- PDE's in 1 (spatial) dimension

  * Heat equation: $\boxed{u_t = c^2 u_{xx}}$

    Need 2 boundary conditions (e.g., $u(0,t) = u(L,t) = 0$)
    and 1 initial condition (e.g., $u(x,0) = h(x)$).

    Solution is of the form, e.g., $\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-c^2 n^2 t}}$

    Has a steady-state solution (heat diffuses).

  * Wave equation: $\boxed{u_{tt} = c^2 u_{xx}}$

    Need 2 boundary conditions (e.g., $u(0,t) = u(L,t) = 0$)
    and 2 initial conditions (e.g., $u(x,0) = h_1(x)$; initial wave
    $u_t(x,0) = h_2(x)$; initial velocity)

    Solution is of the form, e.g., $\boxed{u(x,t) = \sum_{n=1}^{\infty} (a_n \cos cnt + b_n \sin cnt) \sin nx}$

    Has no steady-state solution (waves propogate forever).

This week: PDE's in higher dimensions

In 2 (spatial) dimensions, the heat & wave equations are

* Heat equation $u_t = c^2(u_{xx} + u_{yy})$
* Wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$

Let $u(x_1, \ldots, x_n, t)$ be a function in n (spatial) variables.

The Laplacian of u is $\nabla \cdot \nabla u = \boxed{\nabla^2 u = \sum_{n=1}^{n} \frac{\partial^2 u}{\partial x_i}}$

(Recall that $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, so $\nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$).

In n dimensions, our familiar PDE's are:

* Heat equation: $u_t = c^2 \nabla^2 u$
* Wave equation: $u_{tt} = c^2 \nabla^2 u$

(Note: Sometimes the Laplace operator ∇^2 is written Δ).

Steady-state solutions: Occur for the heat equation, but not for the wave equation (heat diffuses, waves propogate).

Note that "steady-state" means that $u_t = 0$. Solutions to the heat equation approach this steady-state solution, because "eventually, the temperature doesn't change w.r.t. time."

Thus, all steady-state solutions satisfy $0 = u_t = c^2 \nabla^2 u$,

i.e., $\nabla^2 = 0 \Rightarrow \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$.

Def: A function is harmonic if $\nabla^2 u = 0$.

Example: $f(x,y) = x^2 - y^2$ is harmonic:

$f_{xx} = 2$

$f_{yy} = -2$

$\Rightarrow \nabla u = f_{xx} + f_{yy} = 2 - 2 = 0$ ✓

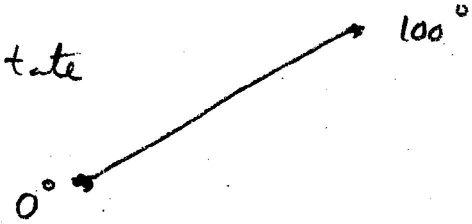
Visualizing harmonic functions:

If $u(x,t)$ is a solution to the heat equation, then

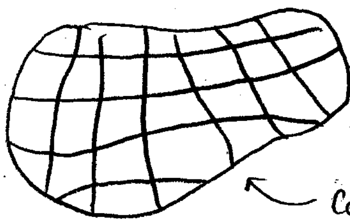
$\lim_{t\to\infty} \frac{\partial u}{\partial t} = 0$. "Temperature will spread out evenly."

Big idea: Steady-state solutions to the heat equations are harmonic functions, and are as "flat as possible."

In 1D: Consider the temperature $u(x,t)$ of a bar with $u(0,t)=0$, $u(L,t)=100$. The steady-state solution satisfies $0 = u_t = c^2 u_{xx}$, so it is a straight line, regardless of initial condition.



Physical interpretation: Stretch out plastic wrap over a bent circular wire, as tight as possible.

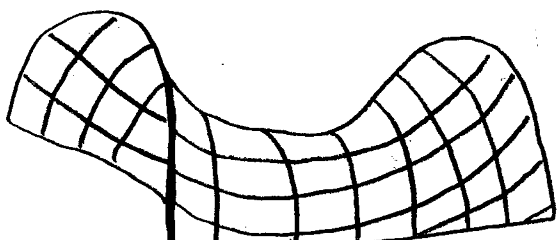


Coat hanger/wire.

* The surface is a harmonic function!

Fact: If f is harmonic, then for any closed bounded region R, f achieves its min & max values on the boundary, ∂R.

Example: $f(x) = x^2 - y^2$

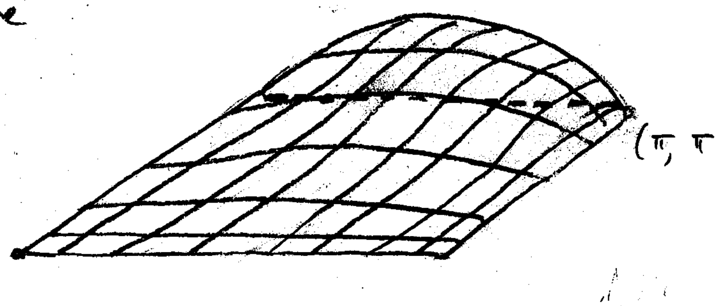


* No local max or mins!

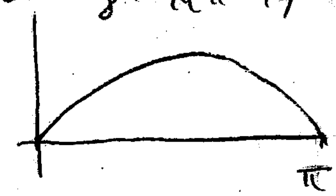
Picture cutting this surface with a "cookie cutter." The max & min points will be on the boundary.

Example 1: Let $u(x,y)$ be a 2-variable function defined for $0 \le x, y \le \pi$ that satisfies the following initial value problem:

$$u_{xx} + u_{yy} = 0$$
$$u(0,y) = u(\pi,y) = u(x,0) = 0$$
$$u(x,\pi) = x(\pi - x)$$



Back edge: $x(\pi - x)$



* Physical situation: $u(x,y)$ is the steady-state solution of a 2D heat equation PDE, where 3 sides are fixed at 0°, and one at $u(x,\pi) = x(\pi - x)$.

Let's solve this (Again, by separation of variables!)

Step 1: Assume $u(x,y) = X(x)\,Y(y)$.

Plug back in: $u_{xx} = X''Y$, $u_{yy} = XY''$

$$\Rightarrow u_{xx} + u_{yy} = X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y}$$

Step 2: Set equal to const λ: $\boxed{\frac{X''}{X} = \frac{-Y''}{Y} = \lambda}$

Get 2 ODEs: $X'' = \lambda X$, $Y'' = -\lambda Y$.

Step 3: Solve these ODEs (with boundary conditions):

$$u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$$
$$u(\pi,y) = X(\pi)Y(y) = 0 \Rightarrow X(\pi) = 0$$
$$u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0$$

Note: The 4^{th} boundary condition $u(x,\pi)$ isn't useful now.

We now have 2 IVPs: (i) $X'' = \lambda X$, $X(0) = X(\pi) = 0$

(ii) $Y'' = -\lambda Y$, $Y(0) = 0$

Let's solve these:

(i) We've done this before: $\boxed{\lambda = -n^2}$, $\boxed{X_n(x) = b_n \sin nx}$

(ii) $Y'' = n^2 Y$, $Y(0) = 0$.

$Y_n(y) = A_n \cosh ny + B_n \sinh ny$ (this will be easier than $C_1 e^{ny} + C_2 e^{-ny}$)

$Y(0) = A_n = 0 \Rightarrow \boxed{Y_n(y) = B_n \sinh ny}$

Step 4: The general solution is $u(x,y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y)$,

i.e, $\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx \sinh ny}$

Step 5: Use the final boundary condition (plug in $y = \pi$):

$$u(x,\pi) = \sum_{n=1}^{\infty} \underbrace{\left(b_n \sinh n\pi\right)} \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \underbrace{\frac{4}{\pi n^3}\left(1-(-1)^n\right)} \sin nx$$

Equate coefficients: $b_n \sinh n\pi = \frac{4}{\pi n^3}\left(1-(-1)^n\right)$

$$\Rightarrow \boxed{b_n = \frac{4\left(1-(-1)^n\right)}{\pi n^3 \sinh n\pi}}$$

Therefore, our (unique) solution to the IVP

$u_{xx} + u_{yy} = 0$, $u(0,y) = u(\pi,y) = u(x,0) = 0$, $u(x,\pi) = x(\pi - x)$

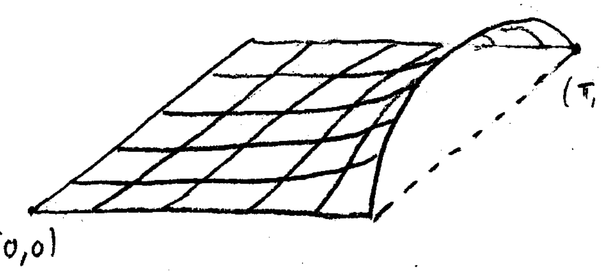
is $\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4\left(1-(-1)^n\right)}{\pi n^3 \sinh n\pi} \sin nx \sinh ny}$

<u>Example 2</u>: Consider the following IVP:

$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = u(x,\pi) = u(0,y) = 0$$

$$u(\pi, y) = y(\pi - y).$$



* This is exactly the same problem as Example 1, but the roles of x & y are reversed!

Thus (by symmetry), the solution is

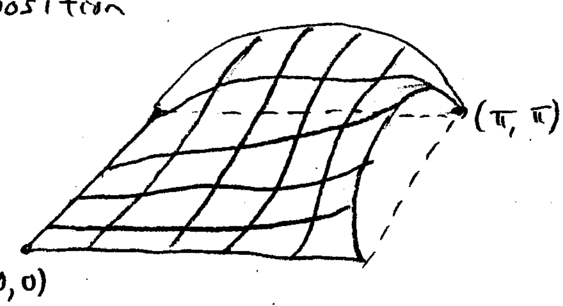
$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sinh nx \, \sin ny}$$

<u>Example 3</u>: The following IVP is a "superposition" of Examples 1 and 2:

$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = u(0,y) = 0$$

$$u(x,\pi) = x(\pi - x), \qquad u(\pi, y) = y(\pi - y)$$



Not surprisingly, the solution to this IVP is the <u>sum</u> of the solutions to Examples 1 & 2:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \left[(\sin nx + \sinh ny) + (\sinh nx + \sin ny) \right]$$

<u>Think</u>: Why does this make sense physically, in terms of steady-state heat distributions?

Heat equation in 2D: $u_t = c^2(u_{xx} + u_{yy})$

Example 4: Let $u(x, y, t)$ = temp. of a square region, $0 \le x, y \le \pi$, subject to

$u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0$ (Boundary fixed at $0°$).

$u(x, y, 0) = 2 \sin x \sin 2y + 3 \sin 4x \sin 5y$ (Initial heat distribution)

Let's solve this!

Step 1: Assume the solution has the form $u(x, y, t) = \underbrace{f(x, y)}_{\text{f'n of position}} \underbrace{g(t)}_{\text{f'n of time}}$

Plug back in: $u_{xx} = f_{xx}\, g$, $u_{yy} = f_{yy}\, g$, $u_t = f g'$

$$u_t = c^2(u_{xx} + u_{yy}) \Rightarrow f g' = c^2 f_{xx}\, g + c^2 f_{yy}\, g$$

$$\Rightarrow \frac{g'}{c^2 g} = \frac{f_{xx} + f_{yy}}{f} = \frac{\nabla^2 f}{f} = \lambda$$

Step 2: Set equal to a const λ (see above), & get 2 equations:

$\boxed{\begin{array}{l} g' = c^2 \lambda g \\ \nabla^2 f = \lambda f \end{array}}$ $\Rightarrow g(t) = C e^{c^2 \lambda t}$

"Helmholz equation"

Step 3: Solve these equations (i.e., the Helmholz equation; we know $g(t)$).

Boundary conditions: $u(0, y, t) = f(0, y)\, g(t) = 0 \Rightarrow f(0, y) = 0$

(we'll need these) $u(\pi, y, t) = f(\pi, y)\, g(t) = 0 \Rightarrow f(\pi, y) = 0$

$u(x, 0, t) = f(x, 0)\, g(t) = 0 \Rightarrow f(x, 0) = 0$

$u(x, \pi, t) = f(x, \pi)\, g(t) \overset{\cdot}{=} 0 \Rightarrow f(x, \pi) = 0$.

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Helmholz equation: $f_{xx} + f_{yy} = \lambda f$.

Separate variables! Assume $f(x,y) = X(x)\,Y(y) \Rightarrow f_{xx} = X''Y, \; f_{yy} = XY''$.

Plug back in: $X''Y + XY'' = \lambda XY \Rightarrow \dfrac{X''Y}{XY} + \dfrac{XY''}{XY} = \lambda$

$$\Rightarrow \boxed{\frac{X''}{X} + \frac{Y''}{Y} = \lambda}.$$

Rewrite as $\underbrace{\dfrac{X''}{X}}_{\text{depends only on } x} = \underbrace{\lambda - \dfrac{Y''}{Y}}_{\text{depends only on } y} = \mu$ ← must be constant!

<u>We get 2 ODEs</u>: $X'' = \mu X, \quad Y'' = \overbrace{(\lambda - \mu)}^{\text{call this } \nu} Y \Rightarrow \lambda - \mu = \nu$

$Y'' = \nu \qquad \Rightarrow \boxed{\lambda = \nu + \mu}$

Recall boundary conditions: $f(0,y) = X(0)\,Y(y) = 0 \Rightarrow X(0) = 0$

$f(\pi, y) = X(\pi)\,Y(y) = 0 \Rightarrow X(\pi) = 0$

$f(x,0) = X(x)\,Y(0) = 0 \Rightarrow Y(0) = 0$

$f(x,\pi) = X(x)\,Y(\pi) = 0 \Rightarrow Y(\pi) = 0$

<u>ODE 1</u>: $X'' = \mu X, \quad X(0) = X(\pi) = 0 \Rightarrow \boxed{X_n(x) = b_n \sin nx, \quad \mu = -n^2}$

<u>ODE 2</u>: $Y'' = \nu Y, \quad Y(0) = Y(\pi) = 0 \Rightarrow \boxed{Y_m(y) = B_m \sin my, \quad \nu = -m^2}$

Recall: $\boxed{\lambda = \nu + \mu = -(n^2 + m^2)}$. Thus, for each pair m & n, we have

solutions: $\boxed{f_{nm}(x,y) = b_{nm} \sin nx \sin my}, \qquad \boxed{g_{nm}(t) = C_{nm} e^{-c^2(n^2+m^2)t}}$

Step 4: The general solution to the heat equation is thus

$$u(x,y,t) = \sum_{n,m\geq 0}^{\infty} f_{nm}(x,y,t)\, g_{nm}(t) \quad \text{(summing over all pairs } n \,\&\, m\text{)}$$

i.e.,
$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{nm} \sin nx \sin my \; e^{-c^2(n^2+m^2)t}}$$

Step 5: Use initial condition (plug in $t=0$):

$$u(x,y,0) = \sum_{n,m\geq 1} b_{mn} \sin nx \sin my = 2\sin x \sin 2y + 3\sin 4x \sin 5y.$$

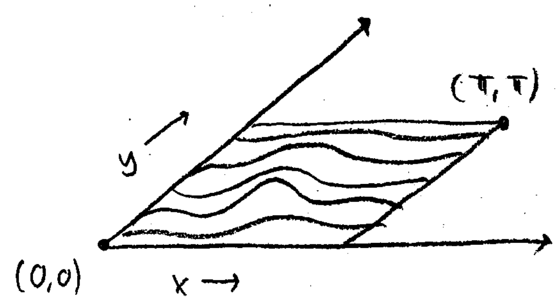
Equate coefficients: $b_{12} = 2$, $b_{45} = 3$, all other $b_{nm} = 0$.

Thus, the (unique) solution to the initial value problem is

$$\boxed{u(x,y,t) = 2\sin x \sin 2y\, e^{-5c^2t} + 3\sin 4x \sin 5y\, e^{-41c^2t}}$$

Wave equation in 2D: $u_{tt} = c^2(u_{xx} + u_{yy})$

Example 5: Let $u(x,y,t)$ = displacement of a point (x,y) on a square membrane of side-length π.

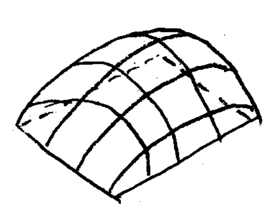


$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0$ (Boundary is immobile)

$u(x,y,t) = p(x)\, q(x)$ Initial wave (displacement)

$u_t(x,y,0) = 0$ Initial wave velocity (vertically).

Let's solve this if $p(x) = x(\pi - x)$
$q(x) = y(\pi - y)$



← Initial wave "paraboloid-like"

* Solving this is almost the same as solving the 2D heat equation.

The only difference in the general sol'n is $g_{nm}(t)$!

Step 1: Assume $u(x,y,t) = f(x,y)\,g(t)$

Plug back in: $f g'' = c^2 g f_{xx} + c^2 g f_{yy}$

$$\Rightarrow \frac{g''}{c^2 g} = \frac{f_{xx}+f_{yy}}{f} = \frac{\nabla^2 f}{f} = \lambda$$

Step 2: Get 2 equations: (i) $\boxed{\nabla^2 f = \lambda f, \quad f(0,y) = f(\pi,y) = f(x,0) = f(x,\pi) = 0}$

(same as for heat equation!)

(ii) $\boxed{g'' = c^2\lambda g}$ (Was $g' = c^2\lambda g$ for heat equation)

We have an init. cond. on g: $u_t(x,y,0) = f(x,y)\,g'(0) = 0 \Rightarrow \boxed{g'(0)=0}$

Step 3: Solve for f & λ: $\boxed{\lambda = -(n^2+m^2)}$

(see 2D Heat eq'n) $\boxed{f_{nm}(x,y) = b_{nm} \sin nx \sinh my}$

Solve for g: $g'' = c^2 \lambda g = -c^2(n^2+m^2) g$

$$\Rightarrow g_{nm}(t) = A_{nm}\cos\left(c\sqrt{n^2+m^2}\,t\right) + B_{nm}\sin\left(c\sqrt{n^2+m^2}\,t\right)$$

$g'(0) = 0 \Rightarrow B_{nm} = 0 \Rightarrow \boxed{g_{nm}(t) = A_{nm}\cos\left(c\sqrt{n^2+m^2}\,t\right)}$

Step 4: Find the general solution.

* For each choice of n & m, we have a solution to the wave equation

$$u_{nm}(x,y,t) = f_{nm}(x,y)\,g_{nm}(t) = b_{nm} \sin nx \sin my \cos\left(c\sqrt{n^2+m^2}\,t\right)$$

Thus, the general solution is

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1} b_{nm} \sin nx \sin my \cos\left(c\sqrt{n^2+m^2}\,t\right)}$$

Note: Alternatively, we can write $\sum_{n,m\geq 1}$ instead of $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$.

Step 5: Use the final initial condition (plug in $t=0$).

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{nm} \sin nx \sin my = \left(\sum_{n=1}^{\infty} B_n \sin nx\right)\left(\sum_{m=1}^{\infty} \beta_m \sin my\right)$$

$$= p(x)\,q(y) = x(\pi - x)\, y(\pi - y)$$

$$= \left(\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \sin nx\right)\left(\sum_{m=1}^{\infty} \frac{4(1-(-1)^m)}{\pi m^3} \sin my\right)$$

Thus, $b_{nm} = B_n \cdot \beta_m = \dfrac{16(1-(-1)^n)(1-(-1)^m)}{\pi^2 n^3 m^3} = \begin{cases} \dfrac{64}{\pi^2 n^3 m^3} & n \,\&\, m \text{ even} \\ 0 & \text{otherwise} \end{cases}$

The unique solution to our IVP is therefore

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16(1-(-1)^n)(1-(-1)^m)}{\pi^2 n^3 m^3} \sin nx \sin my \cos\left(c\sqrt{n^2+m^2}\, t\right)}$$

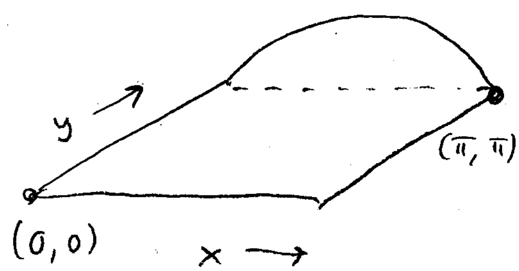
Example 6: 2D Heat equation with non-zero boundary conditions.

$u_t = c^2(u_{xx} + u_{yy})$

$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = 0$

$u(x,\pi,t) = x(\pi - x)$

$u(x,y,0) = x(\pi - x) + \sin x \sin 2y + 3\sin 4x \sin 5y$



This is the the IVP from Example 4 (heat equation with "zero boundary conditions"), but with the boundary conditions from Example 1 (Laplace's equation) added.

* Thus, the unique solution to this IVP is just the sum of the solutions to Examples 1 and 4, i.e.,

$$u(x,y,t) = u_0(x,y,t) + u_{ss}(x,y)$$

$$\boxed{u(x,y,t) = \underbrace{2\sin x \sin 2y\, e^{-5c^2 t} + 3\sin 4x \sin 5y\, e^{-41c^2 t}} + \underbrace{\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sin nx \sinh ny}}$$

"Homogeneous" sol'n; i.e the solution if all four boundary conditions were 0° (sol'n to the IVP in Example 4).

Steady-state sol'n (sol'n to the IVP in Example 1)