MthSc 208, Fall 2010 (Differential Equations)
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HW 11
Due Friday October 1st, 2010

(1) State whether the give system is autonomous or nonautonomous, and also whether it is homogeneous or inhomogeneous.
  (a) $x' = y, \quad y' = x + 4$
  (b) $x' = x + 2y + \sin t, \quad y' = -x + y - \cos t$
  (c) $x' = -2tx + y \quad y' = 3x - y$
  (d) $x' = x + 2y + 4, \quad y' = -2x + y - 3$
  (e) $x' = 3x - y, \quad y' = x + 2y$
  (f) $x' = -x + ty, \quad y' = tx - y$
  (g) $x' = x + y + 4, \quad y' = -2x + (\sin t)y$
  (h) $x' = 3x - 4y, \quad y' = x + 3y$

(2) Transform the given 2$^{\text{nd}}$ initial value problem into an initial value problem of two 1$^{\text{st}}$ order equations (by letting $x_1 = u$ and $x_2 = u'$), and write it in matrix form: $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$, $\mathbf{x}(t_0) = \mathbf{x}_0$. (No need to solve!)
  (a) $u'' + 0.25u' + 4u = 2\cos 3t, \qquad u(0) = 1, \quad u'(0) = -2$
  (b) $tu'' + u' + tu = 0, \qquad u(1) = 1, \quad u'(1) = 0$

(3) In each problem below, an inhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ of two first order ODEs is given. The qualitative behavior of the solutions (as seen through their phase planes) near the critical point are all different. The motivation of this problem is to discover how the eigenvalues and eigenvectors determine the behavior of the solutions. For (a)–(e), carry out the following steps:
  (i) Find the equilibrium solution, or critical point, for the given system.
  (ii) Write the associated homogeneous equation, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and find the eigenvalues and eigenvectors of $\mathbf{A}$.
  (iii) Draw a phase portrait centered at the critical point. (The PPLANE applet at `http://math.rice.edu/~dfield/dfpp.htm` is fantastic.)
  (iv) Describe how solutions of the system behave in the vicinity of the critical point (e.g., do they approach the critial point, depart from it, spiral around it, or something else).
  (a) $x' = -x - 4y - 4, \quad y' = x - y - 6$
  (b) $x' = -0.25x - 0.75y + 8, \quad y' = 0.5x + y - 11.5$
  (c) $x' = -2x + y - 11, \quad y' = -5x + 4y - 35$
  (d) $x' = x + y - 3, \quad y' = -x + y + 1$
  (e) $x' = -5x + 4y - 35, \quad y' = -2x + y - 11$

(4) Tank A contains 10 gallons of a solution in which 5 oz of salt are dissolved. Tank B contains 20 gallons of a solution in which 6 oz of salt are dissolved. Salt water with a concentration of 2 oz/gal flows into each tank at a rate of 4 gal/min. The fully mixed solution drains from Tank A at a rate of 3 gal/min and from Tank B at a rate of 5 gal/min. Solution flows from Tank A to Tank B at a rate of 1 gal/min. Let $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, where $x_1(t)$ (respectively, $x_2(t)$) is the amount of salt in Tank A (resp., Tank B) after time t.
  (a) Write down a system of ODEs (including the initial condition $\mathbf{x}(0)$) that models this situation, and write it in matrix form: $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{c}$.
  (b) What is the steady-state solution, $\mathbf{x}_{ss}$?
  (c) Write down the related homogeneous equation and solve it.
  (d) Find the general solution the orginal system of differential equations modeling the tanks.
  (e) Plug in $t = 0$ and find the particular solution.