MthSc 208, Fall 2010 (Differential Equations)
Dr. Matthew Macauley
HW 15
Due Friday October 22nd, 2010

(1) For each initial value problem, sketch the forcing term, and then solve for $y(t)$. Write your solution as a piecewise function (i.e., not using the Heavysie function). Recall that the function $H_{ab}(t) = H(t-a) - H(t-b)$ is the interval function.
  (a) $y'' + 4y = H_{01}(t), \quad y(0) = 0, \ y'(0) = 0$
  (b) $y'' + 4y = t\, H_{01}(t), \quad y(0) = 0, \ y'(0) = 0$

(2) Define the function
$$\delta_p^\epsilon(t) = \frac{1}{\epsilon}\left(H_p(t) - H_{p+\epsilon}(t)\right) .$$
  (a) Show that the Laplace transform of $\delta_p^\epsilon(t)$ is given by
$$\mathcal{L}\left\{\delta_p^\epsilon(t)\right\} = e^{-sp}\,\frac{1-e^{-s\epsilon}}{s\epsilon} .$$
  (b) Use l'Hôpital's rule to take the limit of the result in part (a) as $\epsilon \to 0$. How does this result agree with the fact that $\mathcal{L}\{\delta_p(t)\} = e^{-sp}$?

(3) Use a Laplace transform to solve the follosing initial value problem:
$$y' = \delta_p(t), \qquad y(0) = 0$$
How does your answer support what engineers like to say, that the "derivative of a unit step is a unit impulse"?

(4) Define the function
$$H_p^\epsilon(t) = \begin{cases} 0, & 0 \le t < p \\ \frac{1}{\epsilon}(x-p), & p \le t < p+\epsilon \\ 1, & t \ge p+\epsilon \end{cases}$$
  (a) Sketch the graph of $H_p^\epsilon(t)$.
  (b) Without being too precise about things, we could argue that $H_p^\epsilon(t) \to H_p(t)$ as $\epsilon \to 0$, where $H_p(t) = H(t-p)$. Sketch the graph of the derivative of $H_p^\epsilon(t)$.
  (c) Compare your result in (b) with the graph of $\delta_p^\epsilon(t)$. Argue that $H_p'(t) = \delta_p(t)$.

(5) Solve the following initial value problems.
  (a) $y'' + 4y = \delta(t), \quad y(0) = 0, \ y'(0) = 0$
  (b) $y'' - 4y' - 5y = \delta(t), \quad y(0) = 0, \ y'(0) = 0$