MthSc 208, Fall 2010 (Differential Equations)
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HW 16
Due Friday October 29th, 2010

(1) Consider the initial value problem

$$y'' + 2y' + 2y = \delta(t), \quad y(0) = y'(0) = 0\,.$$

(a) Use the fact that $\mathcal{L}\{\delta(t)\}(s) = 1$ to show that the solution is $y(t) = e^{-t}\sin t$ for $t \geq 0$.

(b) Show that the solution of

$$y'' + 2y' + 2y = \delta_0^\epsilon(t), \quad y(0) = y'(0) = 0$$

is

$$y_\epsilon(t) = \frac{1}{2\epsilon}\begin{cases} 1 - e^{-t}(\cos t + \sin t), & 0 \leq t < \epsilon; \\ -e^{-t}(\cos t + \sin t) + e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon)), & t \geq \epsilon. \end{cases}$$

Hint: Write $\delta_0^\epsilon(t) = \frac{1}{\epsilon}H_{0\epsilon}(t) = \frac{1}{\epsilon}[H(t) - H(t-\epsilon)]$.

(c) Use l'Hôpital's rule to argue that the solution of part (b) approaches that of part (a) as $\epsilon \to 0$, at least for $t > 0$.

(2) The function

$$f(x) = \begin{cases} 0 & -\pi \leq x < -\pi/2, \\ 1 & -\pi/2 \leq x < \pi/2, \\ 0 & \pi/2 \leq x \leq \pi, \end{cases}$$

can be extended to be periodic of period 2π. Sketch the graph of the resulting function, and compute its Fourier series.

(3) The function

$$f(t) = |x|, \qquad \text{for } x \in [-\pi, \pi]$$

can be extended to be periodic of period 2π. Sketch the graph of the resulting function, and compute its Fourier series.

(4) The function

$$f(x) = \begin{cases} 0 & -\pi \leq x < 0, \\ x & 0 \leq x \leq \pi, \end{cases}$$

can be extended to be periodic of period 2π. Sketch the graph of the resulting function, and compute its Fourier series.

(5) Consider the 2π-periodic function defined by

$$f(x) = \begin{cases} x^2 & -\pi \leq x < \pi, \\ f(x - 2k\pi), & -\pi + 2k\pi \leq x < \pi + 2k\pi. \end{cases}$$

Sketch this function and compute its Fourier series.

(6) Find the Fourier series of the following functions *without* computing any integrals.

(a) $f(x) = 2 - 3\sin 4x + 5\cos 6x$,

(b) $f(x) = \sin^2 x$ [*Hint*: Use a standard trig identity.]

(7) Determine which of the following functions are even, which are odd, and which are neither even nor odd:

(a) $f(t) = x^3 + 3x$.

(b) $f(t) = x^2 + |x|$.

(c) $f(t) = e^x$.

(d) $f(t) = \frac{1}{2}(e^x + e^{-x})$.

(e) $f(t) = \frac{1}{2}(e^x - e^{-x})$.