3. Second order differential equations

We will consider equations of the form \( y'' = f(t, y, y') \).

A solution is any function \( y(t) \) s.t. \( y''(t) = f(t, y(t), y'(t)) \).

Motivating example: \( F = ma \) (Newton’s 2nd law of motion).

Force (could be gravitational, mechanical, etc.) can be a function of time, displacement \( x(t) \), and velocity \( x'(t) \).

\[ F(t, x, x') = mx''(t). \]

Ex. 1: Gravity ("constant" force): \( mx''(t) = -mg \)

Ex. 2: Spring: \( -kx \) (at rest)

Hooke’s law: Restoring force \( R(x) = -kx \) \( \Rightarrow mx''(t) = -kx \)

Think: "Force is proportional to how much we stretch or compress."

Ex. 3: Now, suppose the weight is hanging.

Forces add, so \( F = R(x) + \) (grav. force)

\[ mx'' = -kx + mg \] (Note: Why +mg?)

Ex. 4: Suppose there’s also a damping force (springs never "bounce forever").

This is like air resistance:

- Proportional to velocity
- Acts against the direction of motion.
Thus, \( D(x') = -\mu x' \), \( \mu \) consd.

Forces add, so \( F = D(x') + R(x) + mg \) \( \Rightarrow mx'' = -\mu x' - kx + mg \)

There are 2 "general techniques" for analyzing 2nd order ODE's:

(i) Solving them directly

(ii) Turning them into systems of 1st order ODE's.

Example: \( y'' + 3t y' + 2y = \sin t \), let \( v = y' \), so \( v' = y'' \)

We now have \( \begin{cases} v' + 3tv + 2y = \sin t \\ v = y' \end{cases} \)

We'll do (i) first, because it's an extension of what we've done for 1st order ODE's. (Section 4 will be devoted to (ii); linear systems).

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A linear 2nd order ODE has the form: \( y'' + p(t) y' + q(t) y = g(t) \)

A homogeneous (linear) 2nd order ODE: \( y'' + p(t) y' + q(t) y = 0 \)

**Big idea** The general solution to a linear 2nd order ODE is:

\[ y(t) = C_1 y_1(t) + C_2 y_2(t) + Y_p(t) \]  \( \text{\textit{(a 2-parameter family)} \atop y_h(t)} \)

where \( Y_p(t) \) is any particular solution.

Remark: \( Y_p(t) = 0 \) is a solution of the ODE if and only if it is homogeneous. (Why?)
Examples:

- Find the general soln to \[ y'' = k^2 y \]

  Observe that \( y_1(t) = e^{kt} \) works, as does \( y_2(t) = e^{-kt} \).

  Thus, the general soln is \( y(t) = C_1 e^{kt} + C_2 e^{-kt} \).

- Find the general soln to \[ y'' = -k^2 y \]

  Observe that \( y_1(t) = \cos kt \) works, as does \( y_2(t) = \sin kt \).

  Thus, the general soln is \( y(t) = A \cos kt + B \sin kt \).

- Find the general soln to \[ y'' - 3y' + 2y = 0 \]

  What might be a good guess?

  Try \( y(t) = e^{rt} \) where \( r \) is some constant.

  Solve for \( r \):

  \[
  \begin{align*}
  y &= e^{rt} \\
  y' &= re^{rt} \\
  y'' &= r^2 e^{rt}
  \end{align*}
  \]

  Plug back into \( y'' - 3y' + 2y = 0 \):

  \[
  r^2 e^{rt} - 3re^{rt} + 2e^{rt} = 0
  \]

  \[
  e^{rt}(r^2 - 3r + 2) = 0
  \]

  \[
  e^{rt}(r-1)(r-2) = 0 \implies r = 1 \text{ or } 2
  \]

  Thus, we've found two solns: \( y_1(t) = e^t \), \( y_2(t) = e^{2t} \).

  So the general soln is \( y(t) = C_1 e^t + C_2 e^{2t} \).
Question: What if we have a repeated root?

\[ y'' - 6y' + 9y = 0 \]

e.g., \[ y'' - 6y' + 9y = 0 \]

Again, guess \[ y = e^{rt} \]
\[ y' = re^{rt} \]
\[ y'' = r^2e^{rt} \]
\[ \begin{align*}
  r^2e^{rt} - 6re^{rt} + 9e^{rt} &= 0 \\
  e^{rt}(r^2 - 6r + 9) &= 0 \\
  (r-3)^2 &= 0 \implies r = 3.
\end{align*} \]

We've determined that \[ y_1(t) = C_1 e^{3t} \] is a solution.

But we need one more!

Try \[ y(t) = v(t) e^{3t} \], and solve for \( v(t) \).

If \[ y = v e^{3t} \], then \[ y' = 3e^{3t}v + e^{3t}v' \], and
\[ y'' = 3(3e^{3t}v + e^{3t}v') + (3e^{3t}v' + e^{3t}v'') \]
\[ = 9e^{3t}v + 6e^{3t}v' + e^{3t}v'' \]

Plug back into ODE:
\[ (9e^{3t}v + 6e^{3t}v' + e^{3t}v'') - 6(3e^{3t}v + e^{3t}v') + 9(e^{3t}v) = 0. \]

\[ y'' e^{3t} = 0 \implies v'' = 0 \implies v(t) = C(t + D) \]

Conclusion: \( e^{3t} \) is a sol'n, and \( (C(t+D)) \) is a sol'n for any \( C \neq 0 \), so let's choose \( C = 1 \), \( D = 0 \), so \( v(t) = t \).

Now, \[ y_1(t) = e^{3t} \], \[ y_2(t) = v(t) e^{3t} = t e^{3t} \], so the general solution is \[ y(t) = C_1 e^{3t} + C_2 t e^{3t} \]
Question: What if we have complex roots? i.e., suppose $y'' + py' + qy = 0$, and the roots to the "characteristic equation" are $\Gamma_1, \Gamma_2 = a \pm bi$.

We have 2 solutions: $y_1(t) = e^{(a+bi)t}$, $y_2(t) = e^{(a-bi)t}$.

Thus, the general solution is $y(t) = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t}$.

What do these functions look like??

There's indeed a "better way" to write this general solution.

- Euler's formula: $e^{i\theta} = \cos \theta + i \sin \theta$
- Recall: $\cos(-\theta) = \cos \theta$, $\sin(-\theta) = -\sin \theta$.

$y_1(t) = e^{(a+bi)t} = e^{at} e^{ibt} = e^{at} (\cos bt + i \sin bt)$

$y_2(t) = e^{(a-bi)t} = e^{at} e^{-ibt} = e^{at} (\cos(-bt) + i \sin(-bt))$

$\quad = e^{at} (\cos bt - i \sin bt)$

Remark: Since our ODE is linear and homogeneous, we can
- add two solutions
- multiply a solution by a scalar

... and still have a solution.

Thus, $\frac{1}{2} (y_1(t) + y_2(t)) = e^{at} \cos bt$ is a solution

and $\frac{1}{2i} (y_1(t) - y_2(t)) = e^{at} \sin bt$ is a solution.

Conclusion: The general solution can be written as

$y(t) = A e^{at} \cos bt + B e^{at} \sin bt$, or $y(t) = e^{at} (A \cos bt + B \sin bt)$.
Review: Basic complex numbers & Euler's formula.

In the complex plane, a point \( z \) at a dist. \( R \) from \( 0 \) & angle \( \theta \) is
\[
Re^{i\theta} = R \cos \theta + iR \sin \theta
\]

From this, it is "easy" to see that
\[
\cos(-\theta) = \cos \theta \quad \text{and} \quad \sin(-\theta) = -\sin \theta.
\]
(\( \theta \rightarrow -\theta \) is a reflection across the x-axis. This preserves the x-coordinate but flips the sign of the y-coordinate).

Euler's formula is \( e^{it} = \cos t + i\sin t \), and it implies some neat facts:
\[
\begin{align*}
e^{it} &= \cos t + i\sin t \\
e^{-it} &= \cos t - i\sin t
\end{align*}
\]
\[
\Rightarrow \frac{1}{2}(e^{it} + e^{-it}) = \cos t \\
\frac{1}{2i}(e^{it} - e^{-it}) = \sin t
\]

Also, \( e^{i\pi} = -1 \) \( \Box \) Visually, \( \theta = \pi \) radians.

Another way to see Euler's formula, from Taylor series:
\[
e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad \cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}, \quad \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}
\]
\[
e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \frac{(it)^6}{6!} + \frac{(it)^7}{7!} + \frac{(it)^8}{8!} + \cdots
\]
\[
\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots
\]
\[
\sin t = it - i \frac{t^3}{3!} + i \frac{t^5}{5!} - i \frac{t^7}{7!} + \cdots
\]
\[
\Rightarrow e^{it} = \cos t + i\sin t
\]
Inhomogeneous linear 2nd order ODE: \( y'' + p(t)y' + q(t)y = f(t) \)

* Big idea #1: Suppose a homogeneous ODE has solutions \( y_1(t) \) and \( y_2(t) \). Then \( C_1y_1(t) + C_2y_2(t) \) is a solution as well.

Proof: Plug \( C_1y_1 + C_2y_2 \) back into \( y'' + py' + qy = 0 \):

\[
(C_1y_1 + C_2y_2)'' + p(C_1y_1 + C_2y_2)' + q(C_1y_1 + C_2y_2)
= (C_1y_1'' + p(t)C_1y_1' + q(t)C_1y_1) + (C_2y_2'' + p(t)C_2y_2' + q(t)C_2y_2)
= C_1(y_1'' + p(t)y_1' + q(t)y_1) + C_2(y_2'' + p(t)y_2' + q(t)y_2) = 0.
\]

* Big idea #2: The general solution of a linear ODE has the form \( y(t) = y_h(t) + y_p(t) = C_1y_1(t) + C_2y_2(t) + y_p(t) \), where \( y_p(t) \) is any particular solution, and \( y_h(t) \) solves the related "homogeneous equation." \( y'' + p(t)y' + q(t)y = 0 \).

Proof: Consider \( y(t) - y_p(t) \); the general solution minus \( y_p(t) \):

\[
\frac{y'' + py' + qy = f}{-(y_p'' + py_p' + qy_p = f)}
\]

\[
(y - y_p)'' + p(y - y_p)' + q(y - y_p) = 0
\]

\( y = C_1y_1 + C_2y_2 + y_p \).

How to find a particular solution: "Method of undetermined coefficients."

**Technique:** Guess! (We'll see how to guess right.)

**Example 1:** \( y'' - 5y' + 4y = e^{2t} \), (Homog. eqn: \( y'' - 5y' + 4y = 0 \))
First step: Solve the homog. eqn: \( y_h(t) = C_1 e^{4t} + C_2 e^t \).

Next, guess that there will be a sol'n \( y_p(t) = a e^{2t} \) (why?)

Plug this back in (to solve for \( a \)). Note: \( y_p' = 3ae^{2t}, \quad y_p'' = 9ae^{2t} \)

\[
9ae^{2t} - 5(3ae^{2t}) + 4ae^{2t} = e^{2t}
\]

Combine terms: \(-2ae^{2t} = e^{2t}\) \(\Rightarrow a = -\frac{1}{2}\).

Thus, \( y_p(t) = -\frac{1}{2} e^{2t} \) is a solution!

Using "Big idea #2," \( y(t) = y_h(t) + y_p(t) = C_1 e^{4t} + C_2 e^t - \frac{1}{2} e^{2t} \)

Think: Why did this work?

Ans: Because the forcing term \( f(t) \) and its derivatives had the "same form" (so we could get them to cancel out.)

Example 2: \( y'' + 2y' - 3y = 5 \sin 3t \). (Note: \( y_h(t) = C_1 e^t + C_2 e^{-3t} \))

Problem: If we try \( y_p(t) = a \sin 3t \), then \( y_p(t) = 3a \cos 3t \ldots \)

Not of the "same form" (So they won't cancel)

Fix: Consider a more general particular solution:

\( y_p(t) = a \cos 3t + b \sin 3t \)

\( y_p'(t) = -3a \sin 3t + 3b \cos 3t \)

\( y_p''(t) = -9a \cos 3t - 9b \sin 3t \)

These all "have the same form."
Plugging back in:

\[ y_{p}'' + 2y_{p}' - 3y_{p} = (-9a \cos 3t - 9b \sin 3t) + (-6a \sin 3t + 6b \cos 3t) \]
\[ = (-3a \cos 3t + 3b \sin 3t) \]
\[ = (-12a + 6b) \cos 3t + (-6a - 12b) \sin 3t = 5 \sin 3t \]
\[ = 0 \]

Thus, we have

\[ \begin{align*}
-12a + 6b &= 0 \\
-6a - 12b &= 5
\end{align*} \]
\[ \Rightarrow a = \frac{1}{6}, \quad b = \frac{1}{3} \]
\[ \Rightarrow y_{p}(t) = \frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t \]

The general solution is therefore

\[ y(t) = y_{h}(t) + y_{p}(t) = C_{1}e^{t} + C_{2}e^{-3t} + \left( \frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t \right) \]

Example 3: (polynomial forcing term)

\[ y'' + 2y' - 3y = 6t^{2} + t - 2 \]

Again, \( y_{h}(t) = C_{1}e^{t} + C_{2}e^{-3t} \)

Assume there's a particular solution of the form \( y_{p}(t) = at^{2} + bt + c \)

Why? (Since \( y_{p}'' + 2y_{p}' - 3y_{p} \) will also be a degree-2 poly.)

So, all we have to do is find \( a, b, c \).

Plugging back in:

\[ y_{p}' = 2at + b, \quad y_{p}'' = 2a. \]
\[ y_{p}'' + 2y_{p}' - 3y_{p} = (2a) + 2(2at + b) - 3(at^{2} + bt + c) = 6t^{2} + t - 2 \]
\[ = 6 \]
\[ = 1 \]
\[ = 2 \]

\[ \begin{align*}
-3a &= 6 \\
4a - 3b &= 1 \\
2a + 2b - 3c &= -2
\end{align*} \]
\[ \Rightarrow \begin{align*}
a &= -2 \\
b &= -3 \\
c &= -2/3
\end{align*} \]
\[ \Rightarrow y_{p}(t) = -2t^{2} - 3t - \frac{8}{3} \]
\[ \Rightarrow y(t) = y_{h}(t) + y_{p}(t) \]
\[ y(t) = C_{1}e^{t} + C_{2}e^{-3t} - 2t^{2} - 3t - \frac{8}{3} \]
What could go wrong with this method?
What if the forcing term is a solution to the homogeneous eqn?

**Example 4:** \( y'' - 3y' + 2y = e^{2t} \).

"Characteristic eqn": \( r^2 - 3r + 2 = (r-1)(r-2) \)

\(\Rightarrow y_h(t) = C_1e^t + C_2e^{2t} \).

Assume there's a particular sol'n of the form

\[ y_p(t) = ae^{2t}, \] so \( y_p' = 2ae^{2t}, \) \( y_p'' = 4ae^{2t}. \)

Plug back in: \( (4ae^{2t}) - 3(2ae^{2t}) + 2(ae^{2t}) = e^{2t} \)

\(\Rightarrow 0ae^{2t} = e^{2t} \) no solution for \( a \)!

What happened?

**Note:** \( ae^{2t} \) solves \( y'' - 3y' + 2y = 0 \) (homog. eqn),

thus it will never solve \( y'' - 3y' + 2y = e^{2t} \).

To "fix" this, assume instead that \( y_p(t) = ate^{2t} \)

\( y_p'(t) = 2ate^{2t} + ae^{2t}, \) \( y_p''(t) = 4ate^{2t} + 4ae^{2t} \)

Plug back in: \( (4ate^{2t} + 4ae^{2t}) - 3(2ate^{2t} + ae^{2t}) + 2(ate^{2t}) = e^{2t} \)

\(\Rightarrow Oate^{2t} + ae^{2t} = e^{2t} \)

\(\Rightarrow a = 1 \) \(\Rightarrow y_p(t) = te^{2t} \)

Thus, the general sol'n is \( y(t) = y_h(t) + y_p(t) \)

\[ y(t) = C_1e^t + C_2e^{2t} + te^{2t} \]
Example 5: \( y'' + 2y' - 3y = 5 \sin 3t + 6t^2 + t - 2 \) \((\ast)\)

Again, \( y_h(t) = C_1 e^t + C_2 e^{-3t} \)

Recall that \(-\frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t\) solve \( y'' + 2y' - 3y = 5 \sin 3t \) (Ex. 2)
and \(-2t^2 - 3t - \frac{8}{3}\) solve \( y'' + 2y' - 3y = 6t^2 + t - 2 \) (Ex. 3)

Convince yourself that \( y_p(t) = -\frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t - 2t^2 - 3t - \frac{8}{3} \)
solves \((\ast)\).

Thus, the general solution is \( y(t) = y_h(t) + y_p(t) \), i.e.,

\[
y(t) = C_1 e^t + C_2 e^{-3t} - \frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t - 2t^2 - 3t - \frac{8}{3}
\]

In general (combination forcing terms)

\(\ast\) Suppose \( y'' + py' + qy = f(t) \) has soln \( y_f(t) \)
and \( y'' + py' + qy = g(t) \) has soln \( y_g(t) \).

Then \( y'' + py' + qy = \alpha f(t) + \beta g(t) \) has soln \( \alpha y_f(t) + \beta y_g(t) \).

Application: Harmonic motion.
Recall mass-spring systems.
Let \( x(t) \) = displacement of the mass
Then \( x(t) \) satisfies the following ODE:

\[
Mx'' + 2Cx' + \omega_0^2 x = f(t)
\]

\( c \) = damping constant
\( \omega_0 \) = frequency
\( f(t) \) = driving force.
Example 1: Simple harmonic motion (no damping/damping force)

\[ x'' + kx = 0, \quad k = \omega^2 > 0 \] (Here, \( \omega \) will be "frequency")

\[ x'' = -\omega^2 x \implies x(t) = a \cos \omega t + b \sin \omega t \]

What does this function "look like"?

Here's a trick: We can actually write it as a single cosine wave!

Let's switch to polar coordinates

\[ (a, b) = (A \cos \phi, A \sin \phi) \]

Sneaky little trick:

\[ x(t) = a \cos (\omega t) + b \sin \omega t \]

\[ = A \cos \phi \cos (\omega t) + A \sin \phi \sin (\omega t) \]

\[ = A \cos (\phi - \omega t) \quad \text{Try identity: } \cos (x-y) = \cos x \cos y + \sin x \sin y \]

\[ = A \cos (\omega t - \phi) \]

Big idea: Any function \( x(t) = a \cos (\omega t) + b \sin (\omega t) \) can be written as a single cosine wave, with

* Amplitude \( A = \sqrt{a^2 + b^2} \)
* Phase shift \( \frac{\phi}{\omega} \), where "\( \phi = \tan^{-1} (\frac{b}{a}) \)"

So, \( x(t) = A \cos (\omega t - \phi) = A \cos (\omega (t - \frac{\phi}{\omega})) \)
Note: Since \(-\frac{\pi}{2} < \phi < \frac{\pi}{2}\), \(\phi = \begin{cases} \arctan(b/a) & \text{Q1, Q4} \\ \arctan(b/a) + \pi & \text{Q2} \\ \arctan(b/a) - \pi & \text{Q3} \end{cases}\)

Example: \(X(t) = -3\cos t + 4\sin t\)
\[A = \sqrt{3^2 + 4^2} = 5\]
\[\arctan(-4/3) = -0.927\]

according to your calculator

So, \(\phi = -0.927 + \pi\)

\(\Rightarrow\) \(X(t) = 5 \cos \left[ t - (-0.927 + \pi) \right]\)

Example 2: Simple harmonic motion + external force (grav)

A 2 kg mass is suspended from a spring. The displacement of the spring once the mass is attached is 0.5 m. If the mass is displaced 0.12 m downward from equilibrium, set up and solve the initial value problem that models this.
1st, determine the spring constant: \( k x_0 = mg \).
(at equilibrium, spring force = grav. Force).
\[ k = \frac{mg}{x_0} = \frac{2 \cdot 9.8}{0.5} = 39.2 \text{ N/m} \]

2nd: \( F = m x'' = \Sigma \text{forces} \)

\[ m x'' = -\mu x' - k x + mg \]

acting on the weight \((= 0)\)

\[ m x'' = -k x + mg \]

\[ 2 x'' = -k x + k x_0 = -k(x - x_0) = -39.2 (x - 0.5) \]

\[ 2 x'' + 39.2(x - 0.5) = 0, \quad x(0) = 0.62, \quad x'(0) = 0 \]

Let's solve this: \( 2 x'' + 39.2 x = 19.6 \)

\[ X_h(t) = A \cos \omega t + B \sin \omega t \quad \text{where} \quad \omega = \sqrt{19.6} \]

\[ x_p(t) = 0.5 \quad \text{(set} \ x''=0 \ \text{for} \ x) \]

General solution: \( x(t) = A \cos \omega t + B \sin \omega t + 0.5 \)

Plug in \( x(0) = 0.62 \) \& \( x'(0) = 0 \) \( \leftarrow \) simpler IC. Use it first.

\[ x'(t) = -A \omega \sin \omega t + B \omega \cos \omega t \]

\[ x'(0) = 0 \quad + \quad B \omega = 0 \quad \Rightarrow \quad B = 0 \]
\[ x(t) = A \cos \omega t. \]
\[ x(0) = A = 0.62 \quad \Rightarrow \quad x(t) = 0.12 \cos (\sqrt{19.6} \, t) + 0.5 \]

**Example 3:** Damped harmonic motion. \((c \neq 0)\)

\[ x'' + 2c x' + \omega_0^2 x = 0, \quad c > 0 \]

Assume \(x(t) = e^{rt}\) \(\Rightarrow\) \(r^2 + 2cr + \omega_0^2 = 0\)

\[ r = -c \pm \sqrt{c^2 - \omega_0^2} \]

3 cases:

(i) Complex roots \((c < \omega_0)\): underdamped

\[ x(t) = e^{-ct} (a \cos \omega_0 t + b \sin \omega_0 t) \]

(ii) Double root \((c = 0)\): critically damped

\[ x(t) = C_1 e^{rt} + C_2 t e^{rt} \quad (\text{note: } r_1 = r_2 < 0) \]

(iii) 2 real roots \((c > \omega_0)\): overdamped

\[ x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} \]

\[ \begin{align*}
\text{underdamped} & \quad \text{critically damped} & \quad \text{overdamped}
\end{align*} \]
Example 4: Forced harmonic motion; \( f(t) \neq 0 \).

- Spring attached to a motor
- Source voltage is sinusoidal

\[
X'' + 2cX' + \omega_0^2 X = A \cos \omega t
\]

\[\text{driving frequency}\]
\[\text{natural frequency}\]
\[\text{damping coefficients}\]

Simpler case: No damping, \( c = 0 \)

\[
X'' + \omega_0^2 X = A \cos \omega t
\]

Homog. eqn:

\[
X''_h + \omega_0^2 X = 0
\]

\[
X_h(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.
\]

Case 1: \( \omega \neq \omega_0 \).

\[
X_p(t) = a \cos \omega t + b \sin \omega t.
\]

Need to solve for \( a \) and \( b \).

Plug \( X_p \) back in:

\[
X'' + \omega_0^2 X = A(\omega_0^2 - \omega^2) \cos \omega t + B(\omega_0^2 - \omega^2) \sin \omega t
\]

\[
= A \cos \omega t + 0 \sin \omega t
\]

\[
\Rightarrow \begin{aligned}
A(\omega_0^2 - \omega^2) &= A \\
B(\omega_0^2 - \omega^2) &= 0
\end{aligned}
\]

\[
\Rightarrow a = \frac{A}{\omega_0^2 - \omega^2}, \quad b = 0,
\]

i.e.,

\[
X_p(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t
\]

\( \text{(N.B.}\ A \to \infty \text{ as } \omega_0 \to \omega! \)

General sol'n:

\[
X(t) = X_h(t) + X_p(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t
\]
What does this solution look like?

First, consider equilibrium: \( x(0) = 0 \) \( \Rightarrow \) \( x'(0) = 0 \)

(for simplicity)

\[ C_2 = 0, \quad C_1 = \frac{-A}{w_0^2 - w^2} \]

\[ \Rightarrow x(t) = \frac{A}{w_0^2 - w^2} (\cos wt - \cos w_0 t) \]

- Superposition of waves with different frequencies.

Has anyone experienced this in real life? (Think music!)

Example: \( x(t) = \cos(11t) - \cos(12t) \) (rough sketch)

How to quantify this?

\[ \omega - \Delta \]

\[ \omega_0 \]

\[ \omega \]

\[ w(t) = \frac{\omega + \omega_0}{2} \] (ave. freq.)

and say \( w = \bar{w} - \delta \)

\( \omega_0 = \bar{w} + \delta \)

\[ \Rightarrow x(t) = \frac{A}{w_0^2 - w^2} (\cos wt - \cos w_0 t) = \left( \frac{A \sin \delta t}{2 \bar{w} \delta} \right) \sin \bar{w} t \]

Amplitude is sinusoidal

Case 2: \( w = w_0 \) (Recall: \( f(t) = \cos wt \) is the forcing term)
The ODE is now: \( x'' + w_0^2 x'' = A \cos \omega t \).

But \( x_p(t) = A \cos \omega t \) solves the homog eqn (the "problem case").
So we must try \( x_p(t) = t (a \cos \omega t + b \sin \omega t) \)

Plug \( x_p \) back in:

\[
X_p'' + w_0^2 X_p = \left[ 2w_0(-a \sin \omega t + b \cos \omega t) + \omega_0^2(-a \cos \omega t + b \sin \omega t) \right] \\
+ w_0 t (a \cos \omega t + b \sin \omega t)
\]

\[
= 2w_0(-a \sin \omega t + b \cos \omega t) = A \cos \omega t + 0 \sin \omega t
\]

\[
\begin{align*}
-2w_0 a &= 0 \\
2w_0 b &= A
\end{align*}
\]

\[
\Rightarrow a = 0, \quad b = \frac{A}{2w_0}
\]

Thus \( x_p(t) = \frac{A}{2w_0} t \sin \omega t \)

General soln: \( X(t) = X_h(t) + X_p(t) \)

\[
= C_1 \cos \omega t + C_2 \sin \omega t + \left( \frac{A}{2w_0} t \right) \sin \omega t
\]

Look at the long-term behavior. This wave "blows up"!

Example: Again, consider starting at equilibrium: \( x(0) = 0, \ x'(0) = 0 \).

\[
X(0) = C_1 = 0, \quad x'(t) = C_2 w_0 \cos \omega t + \frac{A}{2} t \cos \omega t + \frac{A}{2w_0} \sin \omega t.
\]

\[
x'(0) = C_2 = 0 \quad \Rightarrow \quad X(t) = \frac{A}{2w_0} t \sin \omega t
\]

Real-life example: Tacoma Narrows Bridge

\( w_0 \sim \omega \); envelope "closes up"
2nd order, non-constant coefficient ODE:

Consider the following: \( x^2 y'' + xy' - y = 0 \), solve for \( y(x) \).

What's a good guess? Try \( y(x) = x^r \). (Why?)

\[
\begin{align*}
    y'(x) &= r x^{r-1}, & y''(x) &= r(r-1) x^{r-2}.
\end{align*}
\]

Plug back in:

\[
\begin{align*}
x^2 y'' + xy' - y &= x^2 \cdot r(r-1) x^{r-2} + x \cdot r x^{r-1} - x^r = 0 \\
x^r (r^2 - r - r + 1) &= 0 \quad \Rightarrow \quad r = \pm 1.
\end{align*}
\]

We have 2 solutions:

\[
y_1(x) = x, \quad y_2(x) = x^{-1}.
\]

\[
y(x) = C_1 x + C_2 x^{-1}
\]

What if \( r \) is complex? Consider \( x^2 y'' + xy' + y = 0 \).

Again, guess \( y(x) = x^r \). \( y' = r x^{r-1}, \quad y'' = r(r-1) x^{r-2} \)

Plug back in... get \( x^r (r^2 + 1) = 0 \quad \Rightarrow \quad r = \pm i \).

Thus \( y_1(x) = x^i \), \( y_2(x) = x^{-i} \) are solutions.

Simplify these:

\[
y_1(x) = x^i = (e^{ln x})^i = e^{i \ln x} = \cos(ln x) + i \sin(ln x)
\]

\[
y_2(x) = x^{-i} = (e^{ln x})^{-i} = e^{-i \ln x} = \cos(ln x) - i \sin(ln x)
\]

\[
\frac{1}{2} (y_1 + y_2) = \cos(ln x)
\]

\[
\frac{1}{2i} (y_1 - y_2) = \sin(ln x)
\]

Distinct solutions!

Thus, our general sol'n is

\[
y(x) = C_1 \cos(ln x) + C_2 \sin(ln x).
\]
Remark: In the more general case when \( r = a + bi \), the solution is
\[
y(x) = C_1 e^{at} \cos(b \ln x) + C_2 e^{at} \sin(b \ln x)
\]

Remark: If \( r \) is a repeated root, then we must assume that \( y(x) = v(x) x^r \), and we'll get \( y(x) = C_1 x^r + C_2 x^r \ln x \)
(work omitted, it's slightly tedious).

Let's make things harder.

Consider \( y'' - 4y' + 12y = 0 \).

What do we assume the solution will be?

**Note:** \( y(x) = x^r \) won't work!

Because if \( y = x^r \), \( y' = r x^{r-1} \), \( y'' = r(r-1) x^{r-2} \),
then \( y'' - 4y' + 12y = r(r-1) x^{r-2} + 4r x^{r-1} + 12 x^r = 0 \)

Maybe try \( y(x) = a x^r + b x^{r-2} \)?

Then, we'll get \( (\quad) x^{r-2} + (\quad) x^{r-2} + (\quad) x^r = 0 \).

This will give us a solution, since we have 3 equations (set coeffs to 0) and 3 unknowns \( (a, b, r) \).

But, it'll only give us one solution (up to scalars).
Better method: Assume \( y(x) = \sum_{n=0}^{\infty} a_n x^n \).

Why? Because most "nice" functions have a Taylor series expansion, so let's find that.

\[
\begin{align*}
y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1}, \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}
\end{align*}
\]

Plug back in: \( \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 4 \sum_{n=0}^{\infty} n a_n x^n + 12 \sum_{n=0}^{\infty} a_n x^n = 0 \) (\#)

Rewrite this so we can combine terms.

Let \( m = n - 2 \):

\[
\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=2}^{\infty} (m+2)(m+1) a_{m+2} x^m
\]

(\#) \( \rightarrow \) \( \sum_{m=2}^{\infty} (m+2)(m+1) a_{m+2} x^m \)

We've shifted the index without changing the series.

Observe (example, for motivation):

If \( f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \)

then \( f'(x) = \sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty} (n+1) x^n \)

Now, switch back to using \( n \) (from \( m \)):

(\#) becomes \( \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4 \sum_{n=0}^{\infty} n a_n x^n + 12 \sum_{n=0}^{\infty} a_n x^n = 0 \)

\[
\sum_{n=0}^{\infty} \left[ (n+2)(n+1) a_{n+2} + (12 - 4n) a_n \right] x^n = 0
\]

Set = 0
\[(n+2)(n+1)a_{n+2} + (12-4n)a_n = 0 \quad \text{for all } n.\]

\[\Rightarrow a_{n+2} = \frac{4(n-3)}{(n+2)(n+1)} a_n. \quad \text{This is a recurrence relation.}\]

**Note:** \(y(0) = a_0\) and \(y'(0) = a_1\).

Choose any \(a_0\). All the even \(a_n\)'s are determined.

Choose any \(a_1\). All the odd \(a_n\)'s are determined.

Thus, we have a 2-parameter infinite family of solutions,
and so \(y(x) = \sum_{n=0}^{\infty} a_n x^n\) is the general solution, given this recurrence.

**Remark:** Since the odd \(a_n\) and even \(a_n\) are independent of each other, we could write,
\(y_1(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}\), \(y_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}\),
and \(y(x) = C_1 y_1(x) + C_2 y_2(x)\).

Let's compute the first few terms (in terms of \(a_0 \& a_1\)).

\[a_2 = \frac{-12}{2} a_0 = -6a_0\]
\[a_3 = \frac{-8}{3!} a_1 = -\frac{4}{3} a_1\]
\[a_4 = \frac{-4}{4!} a_2 = \frac{(-4)(-12)}{4!} a_0 = 2a_0\]
\[a_5 = 0\]
\[a_6 = \frac{(4)(-4)(-12)}{6!} a_0 = \frac{4}{15} a_0\]
\[a_7 = 0\]
\[\vdots\]

and so on...

\[a_n = \frac{(4,0-12)(4,2-12)(4,4-12)\ldots[4(n-2)-12]}{n!} \quad \text{for } n \geq 2\]
Remark: If \( a_0 = 0 \), then \( y(x) = a_1 x + a_3 x^3 \)
\[ = a_1 x - \frac{4}{3} a_1 x^3 = \left[ a_1 \left( x - \frac{4}{3} x^3 \right) \right] \]

This is the only polynomial solution, up to scalars (why?).

Summary: To solve \( y'' - 4xy' + 12y = 0 \), we
- Assumed \( y(x) = \sum_{n=0}^{\infty} a_n x^n \)
- Plugged \( y(x) \) back into the ODE.
- Combined into a single sum \( \sum_{n=0}^{\infty} \left[ \quad \right] x^n = 0 \) \{ shifting of \}
- Set coefficients equal to 0 to get a recurrence: \( a_{n+2} = (\quad) a_n \)

Quick review of power series:

**Def:** A power series centered at \( x_0 \) is a series of the form \( \sum_{n=0}^{\infty} a_n (x-x_0)^n \).

Henceforth, we will only consider power series centered at \( x_0 = 0 \), i.e., \( y(x) = \sum_{n=0}^{\infty} a_n x^n \).

A power series converges at \( x \) if the sequence of partial sums converges. Otherwise, it diverges.

**Example:** \( \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} x^n = e^x \) for all \( x \).
Non-example: \( \lim_{N \to \infty} \sum_{n=0}^{N} (-1)^{n} x^{n} \) diverges for \( x=1 \), because the sequence of partial sums of \( \sum_{n=0}^{N} (-1)^{n} x^{n} = 1 - 1 + 1 - 1 + \ldots \) is 1, 0, 1, 0, 1, 0, ...

**Key Point:** Sometimes a series won't converge everywhere.

**Example:** \( y(x) = \sum_{n=0}^{\infty} x^{n} \) :  
- Converges to \( \frac{1}{1-x} \) if \( |x| < 1 \)  
- Diverges if \( |x| \geq 1 \).

**Def:** The **radius of convergence** is the largest number \( R \) such that if \( |x-x_{0}| < R \), \( \sum_{n=0}^{\infty} a_{n}(x-x_{0})^{n} \) converges for all \( x \), we say \( R = \infty \).

**Examples:** \( y(x) = \sum_{n=0}^{\infty} x^{n} \) has \( R = 1 \)  
\( y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n} \) has \( R = \infty \). (It converges to \( e^{x} \)).

**The Ratio Test,** for computing \( R \). 
\[
R = \lim_{n \to \infty} \left| \frac{a_{n}}{a_{n+1}} \right|, \text{ if this limit exists.}
\]

**Example 1:** Taylor series for \( \ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n}}{n} = x - x^{2} - x^{3} - \ldots \)  
So, \( |a_{n}| = 1 \) for all \( n \geq 1 \) \( \implies \) \( R = \lim_{n \to \infty} \left| \frac{a_{n}}{a_{n+1}} \right| = \frac{1}{1} \implies R = 1 \)

**Example 2:** \( y(x) = \sum_{n=0}^{\infty} \frac{1}{3^{n}} x^{n} \). \( a_{n} = \frac{1}{3^{n}} \), \( R = \lim_{n \to \infty} \left| \frac{\frac{1}{3^{n}}}{\frac{1}{3^{n+1}}} \right| = 3 \implies R = 3 \)

**Example 3:** \( e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n} \). \( a_{n} = \frac{1}{n!} \) \( R = \lim_{n \to \infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| = n+1 \implies R = \infty \)
Regular vs. singular points of ODEs:

**Def:** A function \( f(x) \) is real analytic at \( x_0 \) if
\[
f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n,
\]
for some \( R > 0 \).

i.e., real analytic \( \iff \) has a power series.

**Def:** Consider the ODE \( y'' + p(x)y' + q(x)y = 0 \).

* The point \( x_0 \) is an ordinary point if \( p(x) \) and \( q(x) \) are real analytic at \( x_0 \).

* If \( x_0 \) is not ordinary, then it is a singular point.

  * If \( x_0 \) is singular, then it is regular if \( (x-x_0)p(x) \) and \( (x-x_0)^2q(x) \) are real analytic at \( x_0 \).

**Remark:** In most cases, "real analytic" just means "defined.

  e.g., \( 1/x \) is not real analytic at \( x_0 = 0 \).

**Why we care:**

Theorem of Frobenius: Consider an ODE \( y'' + p(x)y' + q(x)y = f(x) \).

* If \( x_0 \) is an ordinary point, and \( p, q, f \) have
  radii of convergence \( R_p, R_q, R_f \), respectively, then there
  is a power series solution \( y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \), with \( R = \min \{ R_p, R_q, R_f \} \).
If \( x_0 \) is a regular singular point, and \((x-x_0)P(x)\), 
\((x-x_0)^2Q(x)\), and \(f(x)\) have radii of convergence \(R_p, R_a, R_f\) respectively, then there is a generalized power series solution \( y(x) = (x-x_0)^c \sum \limits_{n=0}^{\infty} a_n (x-x_0)^n \), for some constant \(c\) (possibly a fraction, or even complex).

Note: If \( x_0 \) is an irregular singular point, we're out of luck.

Example: Consider \( y'' + x^2y - 4y = 0 \). Thus, \( P(x) = x^2, Q(x) = -4 \).

\( P(x) \) and \( Q(x) \) are real analytic for all \( x_0 \), with radii of conv. \( R = \infty \). Thus, by Frobenius, there is a solution \( y(x) = \sum \limits_{n=0}^{\infty} a_n (x-x_0)^n \), valid for all \( x \) (i.e., \( R = \infty \)).

Example: \( y'' - \frac{x}{1-x^2} y' + \frac{p^2}{1-x^2} y = 0 \), \( (p \text{ is a parameter}) \).

Here, \( P(x) = -\frac{x}{1-x^2}, Q(x) = \frac{1}{1-x^2} \).

Note: \( P(x) \) and \( Q(x) \) are real analytic at \( x=0 \):

\[
Q(x) = \frac{1}{1-(x^2)} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \ldots
\]

\[
P(x) = \frac{-x}{1-x^2} = -x \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n+1} = -x - x^3 - x^5 - x^7 - \ldots
\]

By the ratio test, \( R_p = R_a = 1 \).

Thus, by Frobenius, there is a solution \( y(x) = \sum_{n=0}^{\infty} a_n x^n \) with \( R = 1 \). (This ODE is called Chebyshev's equation).
Example: \( x^5 y'' + y' + y = 0. \)

Write as \( y'' + \frac{1}{x^5} y' + \frac{1}{x^5} y = 0. \) \( p(x) = x^5, \quad q(x) = \frac{1}{x^5}. \)

\( x_0 = 0 \) is an irregular singular point, since \( x \cdot p(x) = \frac{1}{x^4} \) isn't defined at \( x_0 = 0. \)

Frobenius does not guarantee a solution of the form \( y(x) = \sum_{n=0}^{\infty} a_n x^n. \) But we could find one of the form \( \sum_{n=0}^{\infty} a_n (x-1)^n \) if we wanted to. (B/c \( x_0 = 1 \) is regular).

Analogy of regular vs. irregular.

\[
\begin{align*}
f(x) &= \frac{x(x-2)}{(x-2)} \\
g(x) &= \frac{x(x-2)}{(x-2)^2}
\end{align*}
\]

This singularity is "fixable". This singularity is "unfixable".

Example: Solve \( 2xy'' + y' + y = 0. \)

Write as \( y'' + p(x) y' + q(x) y = 0, \) \( p(x) = \frac{1}{2x}, \quad q(x) = \frac{1}{2x}. \)

\( x_0 = 0 \) is a regular singular point, since \( x \cdot p(x) = \frac{1}{2} \) and \( x^2 q(x) = \frac{1}{2} x \) are real analytic (i.e., defined).

By Frobenius, there is a solution of the form 
\[
y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.
\]

We'll find it the same way as before.
\[ y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \quad y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \]

\[ 2xy''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} \]

Plug back into the ODE:

\[ \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \]

\[ = x^r \left[ \sum_{n=0}^{\infty} (2n+2r-1)(n+r) a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n \right] = 0 \]

Shift indices up by one (let \( m = n-1 \), or just do in your head):

\[ = \frac{1}{n+1} \left[ \sum_{n=0}^{\infty} (2n+2r+1)(n+r+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n \right] = 0. \]

\[ = (2r+1) a_0 x^{-1} + \sum_{n=0}^{\infty} \left[ \frac{(2n+2r+1)(n+r+1)}{n+1} a_{n+1} + a_n \right] x^n = 0 \]

Set = 0.

\[ (2r+1) r = 0 \]

"Indicial equation"

\[ r = 0 \text{ or } r = \frac{1}{2} \]

We now have two generalized power series solutions:

\[ r = 0: \quad y_0(x) = \sum_{n=0}^{\infty} a_n x^n, \quad a_{n+1} = -\frac{1}{(2n+1)(n+1)} a_n \]

\[ r = \frac{1}{2}: \quad y_1(x) = \sqrt{x} \sum_{n=0}^{\infty} a_n x^n, \quad a_{n+1} = -\frac{1}{(2n+2)(n+3/2)} a_n \]
Note: This time, choosing $a$, determines every $a_n$ but we still have 2 distinct solutions.

The general solution is $y(x) = A y_0(x) + B y_{1/2}(x)$, where $y_0, y_{1/2}$ are as above.

* The power series method really does come up in practice!!!

- Hermite's differential equation: $y'' - 2xy' + 2py = 0$.
  Used for modeling simple harmonic oscillators in quantum mechanics.

- Legendre's differential equation: $(1-x^2)y'' - 2xy' + \rho(\rho+1)y = 0$.
  Used for modeling spherically symmetric potentials in theory of Newtonian gravitation, and in electricity & magnetism (E&M).

- Bessel's equation: $x^2y'' + xy' + (x^2 - \rho^2)y = 0$.
  Used for analyzing vibrations of a circular drum.

- Chebyshev's equation: $(1-x^2)y'' - xy' + \rho^2y = 0$. 