4. Systems of differential equations

To analyze systems of differential equations, we need to heavily use matrix theory (linear algebra). We'll review this now.

Basic Matrix Algebra:

Matrices add in the "obvious way": $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$

Multiplication is slightly more complicated. (Fact $AB \neq BA$, in general!

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

We can only multiply an $n \times m$ matrix ("n rows, m columns") by an $m \times r$ matrix. The result is an $n \times r$ matrix.

Examples: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$, $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+by+cz \\ dx+ey+fz \end{pmatrix}$

$2\times 2 \quad 2\times 1 = 2\times 1 \qquad\qquad 2\times 3 \quad 3\times 1 = 2\times 1$

What doesn't work: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} d & e \\ f & g \end{pmatrix}$

$2\times 2 \quad 3\times 2 \qquad\qquad 3\times 1 \quad 2\times 2$

Def: An $n \times 1$ matrix is called a vector.

Application of matrices: Systems of linear algebraic equations.

Consider the system: $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$ (x_1 & x_2 are the "unknowns").

We can write this as: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, or $A\vec{x} = \vec{b}$

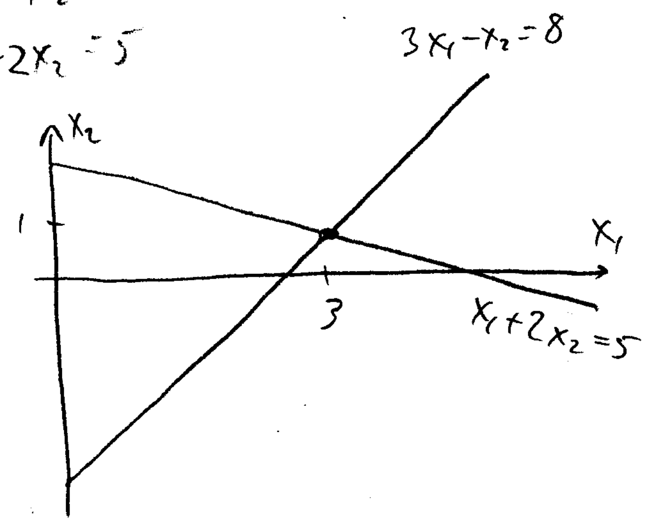
Goal: Solve for $\vec{x}$.

Example 1: Solve the system $\begin{cases} 3x_1 - x_2 = 8 \\ x_1 + 2x_2 = 5 \end{cases}$

This is easy: $x_2 = 3x_1 - 8$

Sub. back in, get $x_1 + 2(3x_1 - 8) = 5$

$x_1 = 3, \quad x_2 = 1$

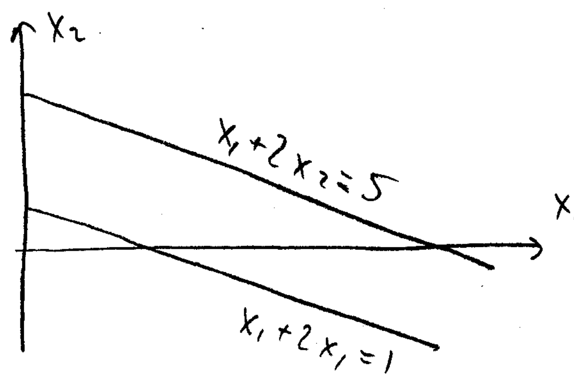


Example 2: Solve the system $\begin{cases} x_1 + 2x_2 = 1 \\ x_1 + 2x_2 = 5 \end{cases}$

$x_1 = 1 - 2x_2$.

Sub. back in, get $1 = 5$, which is false, thus there is no solution.

(i.e, no values of x_1 & x_2 satisfy these equations.)

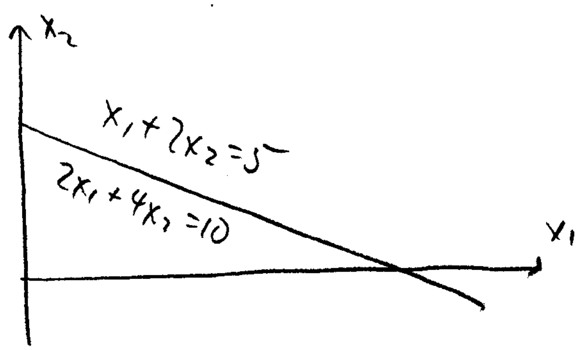


Example 3: Solve the system $\begin{cases} 2x_1 + 4x_2 = 10 \\ x_1 + 2x_2 = 5 \end{cases}$

$x_1 = 5 - 2x_2$

Sub. back in, get $2(5-2x_2) + 4x_2 = 10$, or $10 = 10$

(i.e., any values of x_1 & x_2 satisfy these equations).



Consider the general case: $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

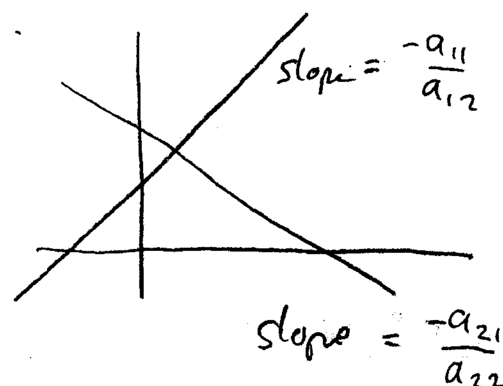
Solving for x_1, we get $x_1 = \dfrac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$, $x_2 = \dfrac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$

Note: There will be a unique solution for x_1 & x_2 if and only if

$a_{11}a_{22} - a_{12}a_{21} \neq 0$.

This quantity is called the determinant of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

and denoted $\det A$, or $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.



Geometric intrepretation:

If the slopes are different, then there will be a unique solution,

and $-\dfrac{a_{11}}{a_{12}} \neq -\dfrac{a_{21}}{a_{22}}$, i.e, $a_{11}a_{22} - a_{12}a_{21} = \det A \neq 0$.

Def: The (2×2) identity matrix is $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

This is the "matrix analog" of the multiplicative identity, 1.

Next, we'll look for a matrix analog of "inverses", i.e., the fact that for any fraction $\frac{a}{b}$: $\frac{a}{b}\cdot\frac{b}{a} = 1$ (the multiplicative identity).

* Given A, we want to find a matrix B such that $AB = BA = I$.

Such a matrix is called the inverse of A, and denoted A^{-1}.

Note that $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11}a_{22}-a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22}-a_{12}a_{21} \end{pmatrix}$

$= (\det A)\cdot I.$

Thus, $A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

When A^{-1} exists, we say that A is <u>invertible</u>, or <u>nonsingular</u>.

Otherwise, A is <u>noninvertible</u>, or <u>singular</u>.

* Thus, A is invertible if and only if $\det A \neq 0$.

<u>Back to systems:</u> We want to solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

If A is invertible, write $A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow \boxed{\vec{x} = A^{-1}\vec{b}}$

<u>Recall example 1:</u> $\begin{cases} 3x_1 - x_2 = 8 \\ x_1 + 2x_2 = 5 \end{cases}$

Write as $\underbrace{\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 8 \\ 5 \end{pmatrix}}_{\vec{b}}$

$\det A = 3\cdot 2 - (-1)\cdot 1 = 7$, so $A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

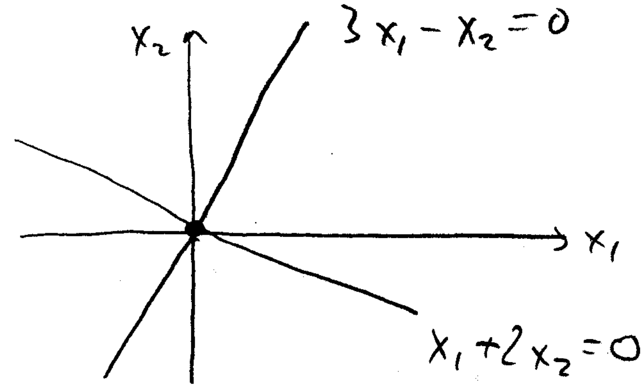
The solution is thus $\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} = \underbrace{\frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}}_{A^{-1}}\underbrace{\begin{pmatrix} 8 \\ 5 \end{pmatrix}}_{\vec{b}} = \frac{1}{7}\begin{pmatrix} 21 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

i.e., $x_1 = 3$, $x_2 = 1$. ✓

Def: If $b_1 = b_2 = 0$ in a system, then the system is <u>homogeneous</u>.

In matrix notation, this is $A\vec{x} = \vec{0}$.

Graphical interpretation: 2 lines through the origin, $\vec{0}$.



<u>Note</u>: If $\det A \neq 0$, then the <u>only</u> solution is $x_1 = x_2 = 0$,

i.e, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Geometrically, this is clear.

Algebraically, $A\vec{x} = \vec{0} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{0} \Rightarrow \vec{x} = \vec{0}$.

<u>A geometric way to view matrices:</u>

A 2×2 matrix is a linear map from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

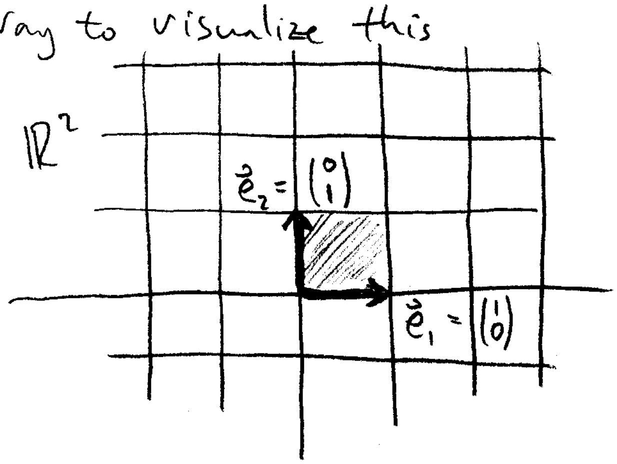
$$A : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \text{i.e, } A\vec{x} = \vec{y}$$

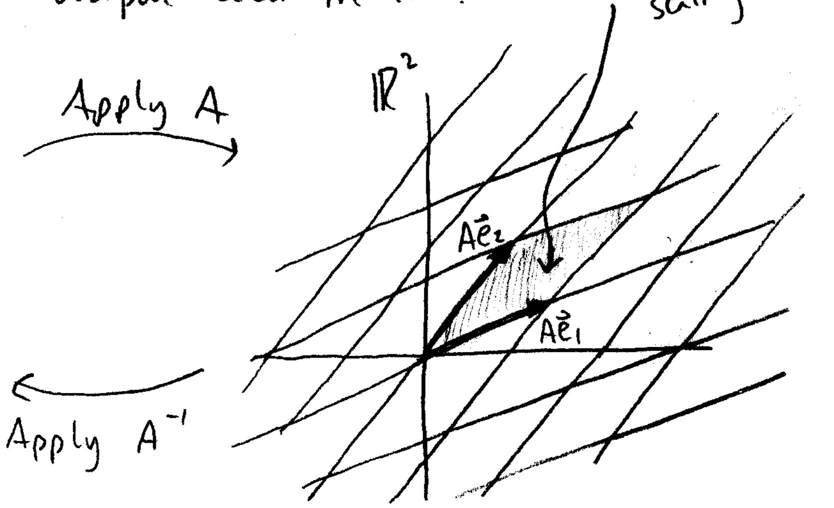
Explicitly, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$

input vector in $\mathbb{R}^2$ ↗ ↖ output vector in $\mathbb{R}^2$.

area = det A "scaling factor"

Way to visualize this





Imagine "picking up $\vec{e}_1$ & $\vec{e}_2$ and moving them to $A\vec{e}_1$ & $A\vec{e}_2$, and the grid "comes along with them."

Note: $A\vec{e}_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, $A\vec{e}_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$

Visually, A^{-1} is the map $\mathbb{R}^2 \to \mathbb{R}^2$ that "undos" A.

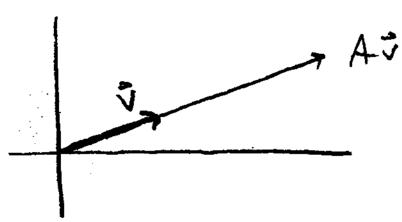
<u>Fact</u>: $\det A$ = area of the image of the unit square under A.

<u>Note</u>: If this area $=0$, then the grid has "collapsed" to 1D, and we can't "undo" (or "pull apart") this. (This is a geometric interpretation of why if $\det A = 0$, then A^{-1} doesn't exist).

<u>Eigenvalues and eigenvectors</u>:

<u>Def</u>: Let A be a matrix. A vector $\vec{v}$ is an <u>eigenvector</u> of A if $A\vec{v} = \lambda\vec{v}$ for some constant λ, which is called an <u>eigenvalue</u>.

Geometrically:



<u>How to find them</u>:

We want to solve $A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} - \lambda\vec{v} = 0$

$$\Rightarrow (A - \lambda I)\vec{v} = 0$$

This equation is homogeneous, so the only solution is $\vec{v} = \vec{0}$, <u>unless</u> $\det(A - \lambda I) = 0$.

So we must solve $\det(A-\lambda I)=0$ for λ.

$$A-\lambda I=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}-\begin{pmatrix} \lambda & 0 \\ 0 & \lambda\end{pmatrix}=\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda\end{pmatrix}$$

$$\det(A-\lambda I)=\underbrace{(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}}_{\text{polynomial in } \lambda}=0$$

$\Rightarrow \lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})=0$ "characteristic polynomial"

<u>Def:</u> The <u>trace</u> of A is the sum of the diagonal entries, and denoted tr A. (<u>Fact</u>: it's also the sum of the eigenvalues!)

If A is 2×2, then $\mathrm{tr}\, A=a_{11}+a_{22}$.

<u>Remark</u>: The characteristic polynomial is $\boxed{\lambda^2-(\mathrm{tr}\, A)\lambda+(\det A)=0}$

The roots of the char. poly. are the <u>eigenvalues</u> of A.

<u>Example 1</u>: (Real, distinct eigenvalues)

Find the eigenvalues of $A=\begin{pmatrix} 1 & 1 \\ 4 & 1\end{pmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda\end{vmatrix}=(1-\lambda)^2-4=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)=0$$

$\Rightarrow \boxed{\lambda_1=3 \text{ and } \lambda_2=-1}$

Now, let's find the associated, $\vec{v}_1$ and $\vec{v}_2$

$\boxed{\lambda_1=3}$: The eigenvector $\vec{v}_1$ solves $(A-3I)\vec{v}=\vec{0}$.

$$(A-3I)\vec{v}=\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3\end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix}=\begin{pmatrix} -2 & 1 \\ 4 & -2\end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix}$$

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i.e, $\begin{cases} -2x_1 + x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{cases}$ This system is <u>redundant</u> (because $\det(A-3I)=0$)

Thus, $-2x_1 + x_2 = 0 \Rightarrow x_2 = 2x_1$

"x_2 is twice as big as x_1"

so, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ 2c \end{pmatrix} = c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

* Any of these is an eigenvector, so let's just pick one: $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Thus, $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A, with eigenvalue $\lambda_1 = 3$.

i.e, $A\vec{v}_1 = 3\vec{v}_1$.

<u>Check</u>: $A\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\vec{v}_1$ ✓

Let's find the other eigenvector

$\boxed{\lambda_2 = -1}$: The eigenvector $\vec{v}_2$ solves $(A+I)\vec{v} = \vec{0}$.

$$(A+I)\vec{v} = \begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ 4x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

i.e, $\begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{cases}$ Again, the equations are redundant, so we just need to consider one of them.

$2x_1 + x_2 = 0 \Rightarrow x_2 = -2x_1$

$\Rightarrow \vec{v} = \begin{pmatrix} c \\ -2c \end{pmatrix} = c\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector for any c.

Let's just pick one, say $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

<u>Check</u>: $A\vec{v}_2 = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -\vec{v}_2$ ✓

* In summary, the matrix A has eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and associated eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$.

Example 2: (Complex eigenvalues)

Find the eigenvalues of $A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{2}-\lambda & 1 \\ -1 & -\frac{1}{2}-\lambda \end{vmatrix} = \left(-\tfrac{1}{2}-\lambda\right)^2 + 1 = \lambda^2 + \lambda + \tfrac{5}{4} = 0$$

$$\Rightarrow \boxed{\lambda_{1,2} = -\tfrac{1}{2} \pm i}$$

Find the eigenvectors:

$\boxed{\lambda_1 = -\frac{1}{2} + i}$: The eigenvector solves $(A - \lambda I)\vec{v} = \vec{0}$

$$(A - (-\tfrac{1}{2}+i)I)\vec{v} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -ix_1 + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{cases}$$

Again, the eq'ns are redundant, and so $-ix_1 + x_2 = 0$.

$\Rightarrow x_2 = i x_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} c \\ ic \end{pmatrix} = c\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector.

$\boxed{\lambda_2 = -\frac{1}{2} - i}$:

$$(A - (-\tfrac{1}{2}-i)I)\vec{v} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{cases}$$

Redundant eq'ns $\Rightarrow ix_1 + x_2 = 0 \Rightarrow x_2 = -ix_1$

$\Rightarrow v_2 = \begin{pmatrix} c \\ -ic \end{pmatrix} = c\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector.

* In summary, the matrix A has eigenvalue $\lambda_1 = -\frac{1}{2} + i$, $\lambda_2 = \frac{1}{2} - i$

and associated eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Example 3: (Repeated eigenvalues, one eigenvector). Consider $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

$\boxed{\lambda = 2}$ (multiplicity 2). Find eigenvector.

Solve $(A - 2I)\vec{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$

Thus $\begin{pmatrix} -c \\ c \end{pmatrix}$ is an eigenvector, so "pick one," say $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We have $\boxed{\lambda = 2, \quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

Example 4: (Repeated eigenvalues, multiple eigenvectors). Consider $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

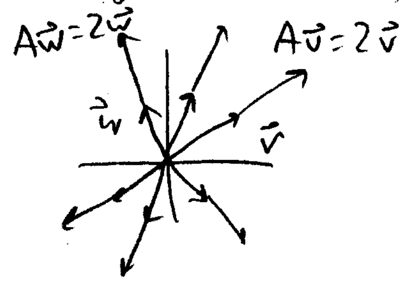
$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 \Rightarrow \lambda = 2 \quad (\text{multiplicity } 2)$$

$\boxed{\lambda = 2}$ Find eigenvectors.

$$(A - 2I)\vec{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 0x_1 + 0x_2 = 0 \\ 0x_1 + 0x_2 = 0 \end{cases}$$

x_1 & x_2 can be anything!

Thus, every vector is an eigenvector (This is quite "unusual")



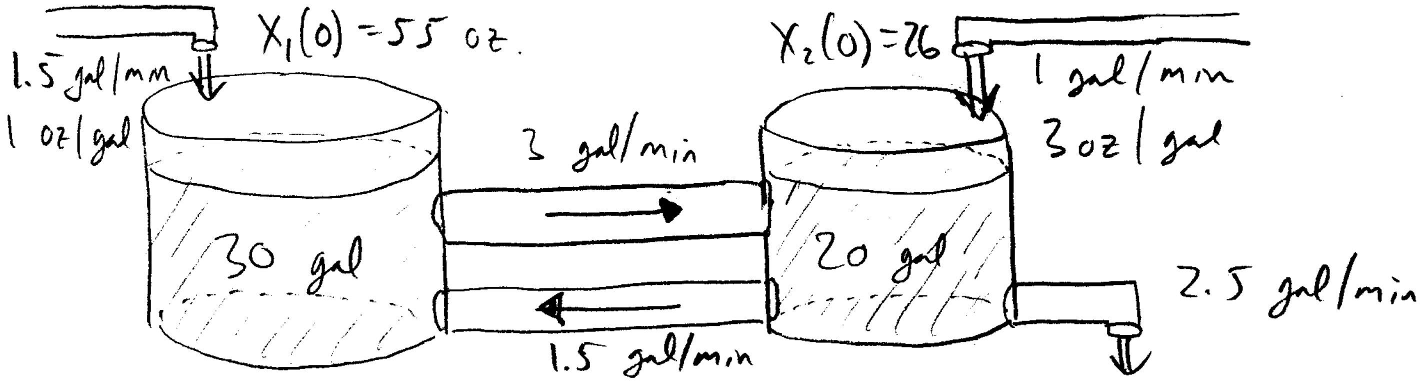
Why is this? Observe that

$$A\vec{v} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 2\vec{v} \text{ for}$$

every vector $\vec{v}$.

Systems of 2 linear 1st order ODEs

Consider the following mixing problem:



Note: Volume of both tanks stays constant.

Let $x_i(t)$ = amt of salt in Tank i, for $i = 1, 2$.

Tank 1: rate in $= \left(1.5\ \frac{gal}{min}\right)\left(\frac{x_2(t)\ oz}{20\ gal}\right) + \left(1.5\ \frac{gal}{min}\right)\left(1\ \frac{oz}{gal}\right) = 0.075x_2 + 1.5$

rate out $= \left(3\ \frac{gal}{min}\right)\left(\frac{x_1(t)\ oz}{30\ gal}\right) = 0.1\,x_1$

Tank 2: rate in $= \left(3\ \frac{gal}{min}\right)\left(\frac{x_1(t)\ oz}{30\ gal}\right) + \left(1\ \frac{gal}{min}\right)\left(3\ \frac{oz}{gal}\right) = 0.1x + 3$

rate out $= \left(4\ \frac{gal}{min}\right)\left(\frac{x_2(t)\ oz}{20\ gal}\right) = 0.2x_2$.

We get a system $\quad x_1' = -0.1x_1 + 0.075x_2 + 1.5, \quad x_1(0) = 55$

("rate in" - "rate out") $\quad x_2' = 0.1x_1 - 0.2x_2 + 3, \quad x_2(0) = 26$

Matrix notation: $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 1.5 \\ 3 \end{pmatrix}}_{\vec{b}}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 55 \\ 26 \end{pmatrix}$

or just $\quad \vec{x}' = A\vec{x} + \vec{b}, \quad \vec{x}(0) = \begin{pmatrix} 55 \\ 26 \end{pmatrix}$.

The variables x_1, x_2 are the state variables

The vector $\vec{x} = x_1\hat{\imath} + x_2\hat{\jmath} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the state vector of the system.

The x_1x_2-plane is the state plane, or phase plane.

Note: $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a curve in the phase plane.

(Think: What does it represent?)

Question: Does this system have a steady-state solution?

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Ans: Yes! It is <u>autonomous</u>, i.e, neither A nor $\vec{b}$ depend on t (in $\vec{x}' = A\vec{x} + \vec{b}$).

Thus, to find a constant, or <u>steady-state</u> solution, set $\vec{x}' = \vec{0}$ and solve for $\vec{x}$:

$$\vec{0} = \vec{x}' = A\vec{x} + \vec{b} \Rightarrow A\vec{x} + \vec{b} = \vec{0} \Rightarrow$$
$$\Rightarrow A\vec{x} = -\vec{b} \Rightarrow \boxed{\vec{x} = -A^{-1}\vec{b}}$$

$\det A = \frac{1}{80}$, so $A^{-1} = \frac{1}{1/80}\begin{pmatrix} -0.2 & -0.075 \\ -0.1 & -0.1 \end{pmatrix} = \begin{pmatrix} -16 & -6 \\ -8 & -8 \end{pmatrix}$

$\Rightarrow \vec{x}_{ss} = -A^{-1}\vec{b} = \begin{pmatrix} -16 & -6 \\ -8 & -8 \end{pmatrix}\begin{pmatrix} -1.5 \\ -3 \end{pmatrix} = \begin{pmatrix} 42 \\ 36 \end{pmatrix}$ is the steady-state sol'n

Now, let's change variables: let $y_1(t) = x_1(t) - 42$
$y_2(t) = x_2(t) - 36$

$\Rightarrow x_1 = y_1 + 42, \quad x_2 = y_2 + 36, \quad x_1' = y_1', \quad x_2' = y_2'.$

Plug back into our system $\begin{cases} x_1' = -0.1x_1 + 0.075x_2 + 1.5, & x_1(0) = 55 \\ x_2' = 0.1x_1 + 0.2x_2 + 3, & x_2(0) = 26 \end{cases}$

and we get a <u>homogeneous</u> system $\begin{cases} y_1' = -0.1y_1 + 0.075y_2, & y_1(0) = 13 \\ y_2' = 0.1y_1 + 0.2y_2 & y_2(0) = -10 \end{cases}$

i.e, $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & 0.2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

$\vec{y}' = A\vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

* Now, we just have to solve the <u>homogeneous</u> system $\vec{y}' = A\vec{y}$, and then add $\vec{x}_p(t) = \begin{pmatrix} 42 \\ 36 \end{pmatrix}$ back to it.

<u>Remark</u>: We're really just doing $\boxed{\vec{x} = \vec{x}_h + \vec{x}_p}$ here!

Next, we'll see how to solve $\vec{y}' = A\vec{y}$.

<u>Solving a homogeneous system of ODE's</u>

Consider a simple example: $\vec{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

This is the system: $\begin{cases} x_1' = -x_1 & x_1(0) = 2 \\ x_2' = -4x_2 & x_2(0) = 3 \end{cases}$

This is easy! The general solution is $x_1(t) = C_1 e^{-t}$, $x_2(t) = C_2 e^{-4t}$.

Writing this in "vector notation," we get

$$\vec{x} = \begin{pmatrix} C_1 e^{-t} \\ C_2 e^{-4t} \end{pmatrix} = C_1 \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ e^{-4t} \end{pmatrix} = \boxed{C_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Plug in initial conditions: $\vec{x}(0) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\Rightarrow C_1 = 2, \quad C_2 = 3 \quad \Rightarrow \quad \boxed{\vec{x}(t) = 2e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

<u>Remark</u>: The eigenvalues & eigenvectors of $\begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$ are

$\lambda_1 = -1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = -4$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Suppose $\vec{v}$ is an eigenvector of A with eigenvalue λ.

* <u>Claim</u>: $\vec{x}(t) = e^{\lambda t} \vec{v}$ solves $\vec{x}' = A\vec{x}$.

<u>Proof</u>: (Easy!) Plug it in and check.

If $\vec{x}(t) = e^{\lambda t}\vec{v} = \begin{pmatrix} e^{\lambda t} x_1 \\ e^{\lambda t} x_2 \end{pmatrix}$, then $\vec{x}'(t) = \begin{pmatrix} \lambda e^{\lambda t} x_1 \\ \lambda e^{\lambda t} x_2 \end{pmatrix} = \lambda e^{\lambda t}\vec{v}$

equal!

Also, $A\vec{x} = A(e^{\lambda t}\vec{v}) = e^{\lambda t}(A\vec{v}) = e^{\lambda t}(\lambda \vec{v}) = \lambda e^{\lambda t}\vec{v}$.

Remark: Since $\vec{x}' = A\vec{x}$ is homogeneous, then for any 2 solutions $\vec{x}_1, \vec{x}_2$, the vector $c_1\vec{x}_1 + c_2\vec{x}_2$ is also a sol'n ("superposition").

Conclusion: Suppose A has distinct eigenvectors $\lambda_1 \neq \lambda_2$ with eigenvectors $\vec{v}_1, \vec{v}_2$. Then the general solution to $\vec{x}' = A\vec{x}$ is $\boxed{\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}$

Not surprisingly, the initial condition $\vec{x}(0) = \vec{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$ determines a unique particular solution.

Example 1a: (Revisited; distinct real (negative) eigenvalues).

$$\vec{y}' = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & 0.2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \vec{y}(0) = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$$

It is easily verified that the eigenvalues & eigenvectors are

$$\lambda_1 = -0.25, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \lambda_2 = -0.05, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Thus, the general solution is $\boxed{\vec{y}(t) = C_1 e^{-0.25t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-0.05t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$

Plug in initial conditions: $\vec{y}(0) = C_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$,

i.e., $\begin{cases} C_1 + 3C_2 = 13 \\ -2C_1 + 2C_2 = -10 \end{cases}$ or just $\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

This has solution $C_1 = 7$, $C_2 = 2$

Thus, we get a particular solution $\boxed{\vec{y}(t) = e^{-0.25t}\begin{pmatrix}7\\-14\end{pmatrix} + e^{-0.05t}\begin{pmatrix}6\\4\end{pmatrix}}$

i.e., $\vec{y}(t) \begin{pmatrix}y_1(t)\\y_2(t)\end{pmatrix} = \begin{pmatrix}7e^{-0.25t} + 6e^{-0.05t}\\-14e^{-0.25t} + 4e^{-0.05t}\end{pmatrix} \begin{matrix}\leftarrow y_1\\ \leftarrow y_2\end{matrix}$

$\Rightarrow \vec{x}(t) = \vec{y}(t) + \begin{pmatrix}42\\36\end{pmatrix} = \begin{pmatrix}7e^{-0.25t} + 6e^{-0.05t} + 42\\-14e^{-0.25t} + 4e^{-0.05t} + 36\end{pmatrix} \begin{matrix}\leftarrow x_1(t)\\ \leftarrow x_2(t)\end{matrix}$

* We'll plot the <u>phase plot</u> of $y(t)$: <u>y_2 vs. y_1.</u>

($\vec{y}(t)$ is easier; it's centered at $\vec{0}$)

Then we'll just shift $\vec{y}(t)$ back to $(42, 36)$, or equivalently, add back the particular (steady-state) sol'n, $\vec{x}_p(t) = \begin{pmatrix}42\\36\end{pmatrix}$.

How do we plot $\vec{y}(t) = C_1\vec{y}_1(t) + C_2\vec{y}_2(t)$? (On the y_2y_1-plane).

Let's try a few "special cases".

<u>Suppose $C_2 = 0$:</u> Then $\vec{y}(t) = C_1\vec{y}_1(t) = \begin{pmatrix}C_1 e^{-0.25t}\\-2C_1 e^{-0.25t}\end{pmatrix} = \begin{pmatrix}y_1\\y_2\end{pmatrix}$

$\downarrow$

<u>Note</u>: The y_1-value is $C_1 e^{-0.25t}$

The y_2-value is $-2C_1 e^{-0.25t}$ $\qquad \vec{0}$ as $t \to \infty$

* Thus, $\vec{y}(t)$ is a curve that has slope $\frac{y_2}{y_1} = -2$ for all t,

i.e., $\vec{y}(t)$ lies on the line $y_2 = -2y_1$.

<u>Suppose $C_1 = 0$:</u> Then $\vec{y}(t) = C_2\vec{y}_2(t) = \begin{pmatrix}3C_2 e^{-0.05t}\\2C_2 e^{-0.05t}\end{pmatrix} \to \vec{0}$ as $t \to \infty$

This curve has slope $\frac{y_2}{y_1} = \frac{2}{3}$ for all t, i.e., it lies on the line $y_2 = \frac{2}{3}y_1$.

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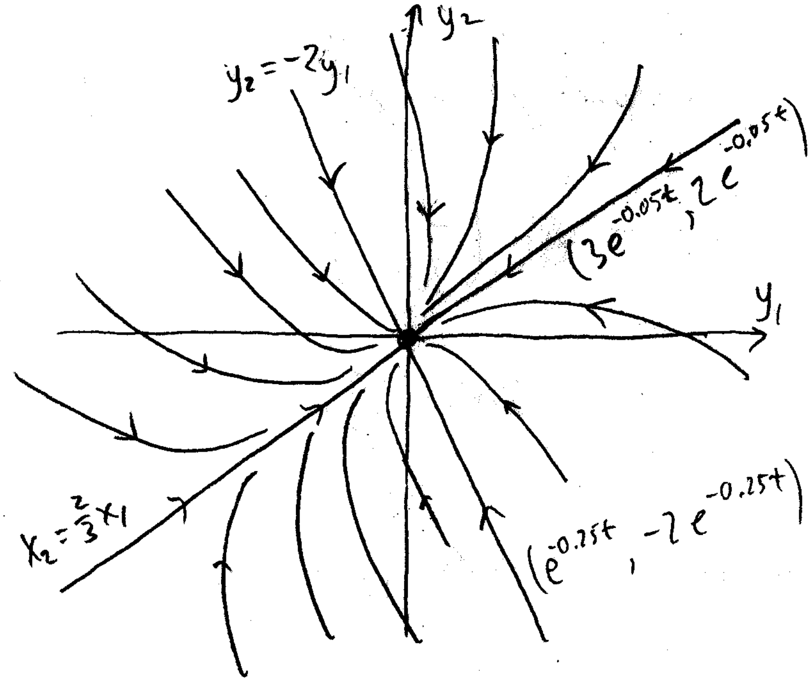
Since $\lim_{t\to\infty} y_1(t) = \vec{0}$ and

$\lim_{t\to\infty} y_1(t) = \vec{0}$, the solution curves on the lines $y_2 = -2y_1$ and $y_2 = \frac{2}{3}y_1$ move towards the origin (the steady-state solution).



Phase plot

Remark: Since $e^{-0.25t} \to 0$ faster than $e^{-0.05} \to 0$, the solution curves "bend" (see phase plot above)

Example 1b (Real eigenvalues, opposite sign),

Consider $\vec{x}' = A\vec{x}$, $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$. Recall: $\lambda_1 = 3$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = -1$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

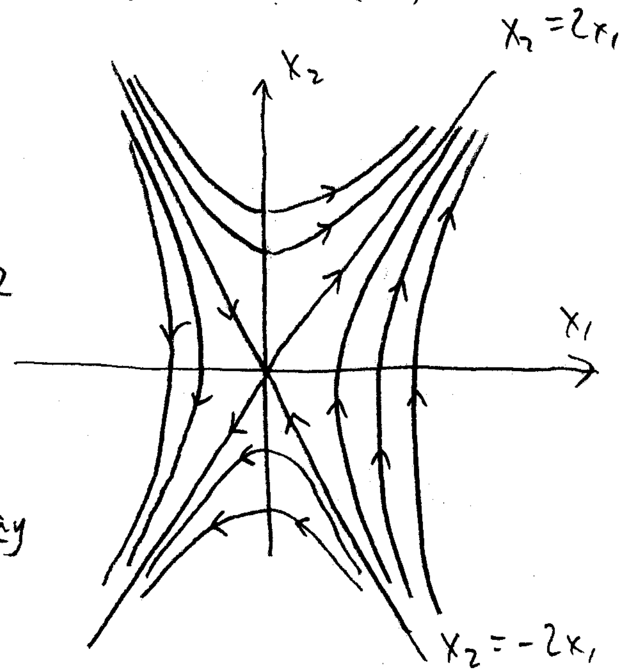
Thus, the general solution is $\vec{x}(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

If $C_2 = 0$: $\vec{x}(t) = \begin{pmatrix} C_1 e^{3t} \\ 2C_1 e^{3t} \end{pmatrix} \Rightarrow$ slope $= \frac{x_2}{x_1} = 2$

and $\lim_{t\to\infty} |\vec{x}(t)| = \infty$.

If $C_1 = 0$: $\vec{x}(t) = \begin{pmatrix} C_1 e^{-t} \\ -2C_1 e^{-t} \end{pmatrix} \Rightarrow$ slope $\frac{x_2}{x_1} = -2$

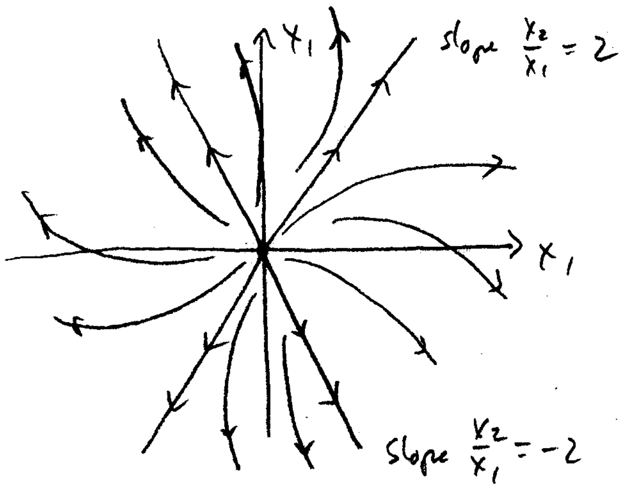
and $\lim_{t\to\infty} |\vec{x}(t)| = 0$.



* Curves on the line $x_2 = 2x_1$ move away from the origin, and those on $x_2 = -2x_1$ move toward the origin.

Example 1c: (Real distinct positive eigenvalues).

Sketch the phase plot for $\vec{x}' = A\vec{x}$, if $\lambda_1 = 1$, $\lambda_2 = 5$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.



* $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow$ solution curve along the line with slope $\frac{x_2}{x_1} = 2$.
* $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow$ solution curve along the line with slope $\frac{x_2}{x_1} = -2$

Remarks:
- The solution curves move away from the origin because $\lim_{t\to\infty} e^t = \infty$ and $\lim_{t\to\infty} e^{5t} = \infty$.
- The curves "bend" because $e^{5t} \to \infty$ faster than $e^t \to \infty$
- If $\lambda_1 = \lambda_2$ but with these same eigenvectors, the phase plot would be star-shaped (no "bend").

Summary (when $\lambda_1 \neq \lambda_2$ are real):

If $\vec{x}' = A\vec{x}$ and A has real eigenvalues $\lambda_1 \neq \lambda_2$ & eigenvectors $\vec{v}_1, \vec{v}_2$:

The general solution is $\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$.

We can plot the "phase portrait" (x_2 vs. x_1) by first drawing the lines with constant slope $\frac{x_2}{x_1}$, for each eigenvector $\vec{v}_i = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

- If $\lambda > 0$, the sol'ns move away from $\vec{0}$ because $\lim_{t\to\infty} |Ce^{\lambda t}\vec{v}| = \infty$.
- If $\lambda < 0$, the sol'ns move towards $\vec{0}$ because $\lim_{t\to\infty} |Ce^{\lambda t}\vec{v}| = \vec{0}$.

The solution curves "bend" depending on how different λ_1 & λ_2 are.

Example 2a: (Complex eigenvalues). Solve $\vec{x}' = \underbrace{\begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}}_{\text{call this } A} \vec{x}$

Recall that $\lambda_1 = -\frac{1}{2} + i \qquad \lambda_2 = -\frac{1}{2} - i$

$\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

As before, we have 2 solutions: $\vec{x}_1(t) = e^{(-\frac{1}{2}+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $x_2(t) = e^{(-\frac{1}{2}-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

but we'd prefer to write these using sines & cosines.

Recall Euler's formula: $e^{it} = \cos t + i \sin t$

$$\vec{x}_1(t) = e^{-\frac{1}{2}t} e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-\frac{1}{2}t}(\cos t + i \sin t)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix}}_{\vec{u}(t)} + i \underbrace{\begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}}_{\vec{w}(t)} = \vec{u}(t) + i\,\vec{w}(t)$$

Claim: Since $\vec{x}_1(t) = \vec{u}(t) + i\vec{w}(t)$ solve $\vec{x}' = A\vec{x}$, so do $\vec{u}(t)$ & $\vec{w}(t)$.

Proof: $\vec{x}_1' = \vec{u}' + i\vec{w}'$ and $A\vec{x}_1 = A(\vec{u} + i\vec{w}) = A\vec{u} + iA\vec{w}$

Since $\vec{u}' + i\vec{w}' = A\vec{u} + iA\vec{w}$, then $\underbrace{\vec{u}' = A\vec{u}}_{\text{real part}}$ and $\underbrace{\vec{w}' = A\vec{w}}_{\text{imaginary part}}$ ✓

Since $\vec{u}(t)$ and $\vec{w}(t)$ are 2 distinct sol'ns (not scalar multiples of each other), the general solution is $\vec{x}(t) = C_1\vec{u}(t) + C_2\vec{w}(t)$,

i.e., $$\vec{x}(t) = C_1\begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + C_2\begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}$$

Remark: Recall the "1D case", solving $y'' + py' + qy = 0$ and $r = a \pm bi$.

We had a solution $x_1(t) = e^{at}e^{ibt} = e^{at}(\cos bt + i \sin bt) = \underbrace{e^{at}\cos bt}_{u(t)} + i\underbrace{e^{at}\sin bt}_{w(t)}$

and the general sol'n was $x(t) = C_1 u(t) + C_2 w(t)$

$$= e^{at}(A\cos bt + B\sin bt).$$

This is really the same thing!

Let's plot the solution:

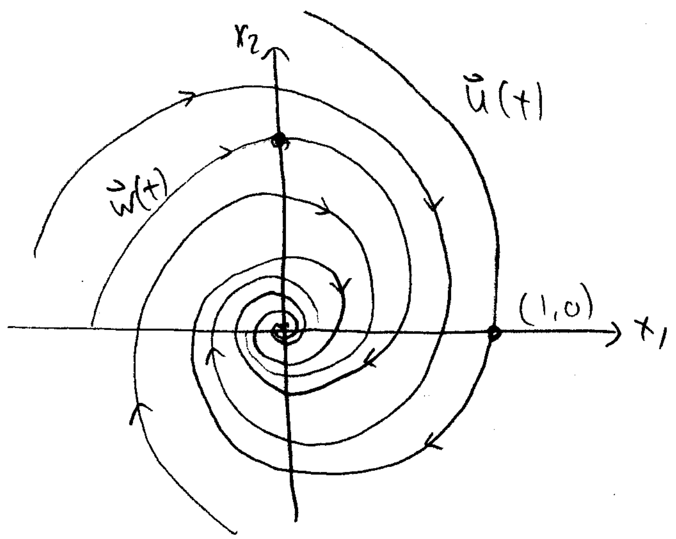
$$\vec{x}(t) = C_1 e^{-t/2}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + C_2 e^{-t/2}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

Note: The curves $\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$ and $\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$ are clockwise circles. (Why?)

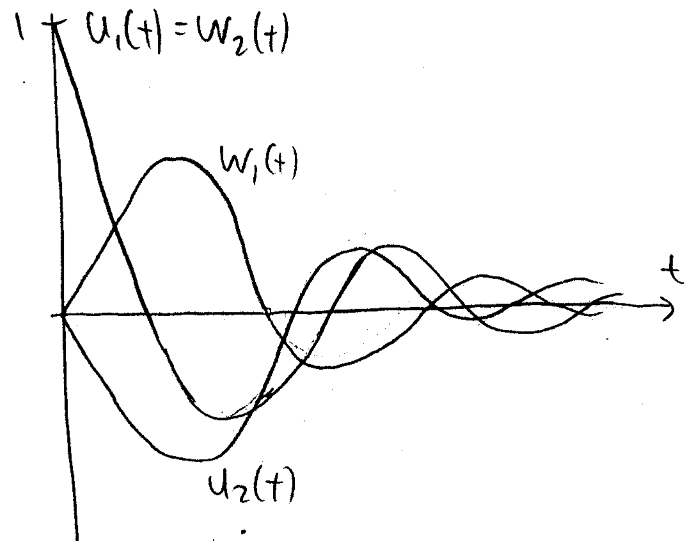
(In vector calc., we wrote these as, e.g., $c(t) = (\cos t, \sin t)$)

Since $\lim_{t\to\infty} e^{-t/2} = 0$, $\vec{u}(t) = e^{-t/2}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$ is an inward spiral, starting at $(1,0)$

and $\vec{w}(t) = e^{-t/2}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$ is an inward spiral, starting at $(0,1)$



Phase plot



Component plots

Note: If $\lambda_{1,2} = a \pm bi$, with $a > 0$, the the phase plot would consist of outward spirals. (Think: what if $a = 0$? This comes next...)

Example 26: (Purely imaginary eigenvalues)

Consider the system $\vec{x}' = \begin{pmatrix}1/2 & -5/4\\ 2 & -1/2\end{pmatrix}\vec{x}$, $\vec{x}(0) = \begin{pmatrix}-1\\ -2\end{pmatrix}$.

Check: $|A - \lambda I| = \lambda^2 + \frac{9}{4} \Rightarrow \lambda_1 = \frac{3}{2}i$, $\lambda_2 = -\frac{3}{2}i$, and $\vec{v}_1 = \begin{pmatrix}5\\ 2-6i\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}5\\ 2+6i\end{pmatrix}$.

We have 2 solutions: $\vec{x}_1(t) = e^{\frac{3}{2}it}\begin{pmatrix} 5 \\ 2-6i \end{pmatrix}$, $\vec{x}_2(t) = e^{-\frac{3}{2}it}\begin{pmatrix} 5 \\ 2+6i \end{pmatrix}$.

Apply Euler's formula, & separate into real & imaginary parts $\left(\vec{x}_1(t) = \vec{u}(t) + i\vec{v}(t)\right)$.

$$\vec{x}_1(t) = \left(\cos \tfrac{3}{2}t + i \sin \tfrac{3}{2}t\right)\begin{pmatrix} 5 \\ 2-6i \end{pmatrix} \qquad \text{(multiplying through \& collecting terms...)}$$

$$= \underbrace{\begin{pmatrix} 5\cos\frac{3}{2}t \\ 2\cos\frac{3}{2}t + 6\sin\frac{3}{2}t \end{pmatrix}}_{\vec{u}(t)} + i \underbrace{\begin{pmatrix} 5\sin\frac{3}{2}t \\ 2\sin\frac{3}{2}t - 6\cos\frac{3}{2}t \end{pmatrix}}_{\vec{w}(t)} = \vec{u}(t) + i\,\vec{w}(t).$$

Thus, the general solution is

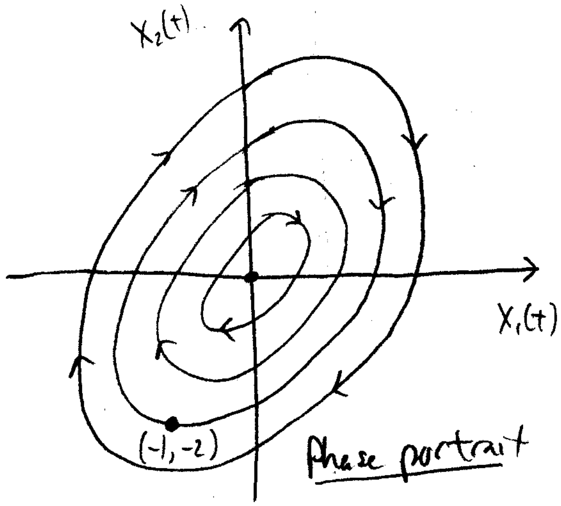
$$\vec{x}(t) = C_1\begin{pmatrix} 5\cos\frac{3}{2}t \\ 2\cos\frac{3}{2}t + 6\sin\frac{3}{2}t \end{pmatrix} + C_2\begin{pmatrix} 5\sin\frac{3}{2}t \\ 2\sin\frac{3}{2}t - 6\cos\frac{3}{2}t \end{pmatrix}.$$

Let's plot the curve satisfying $\vec{x}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$:

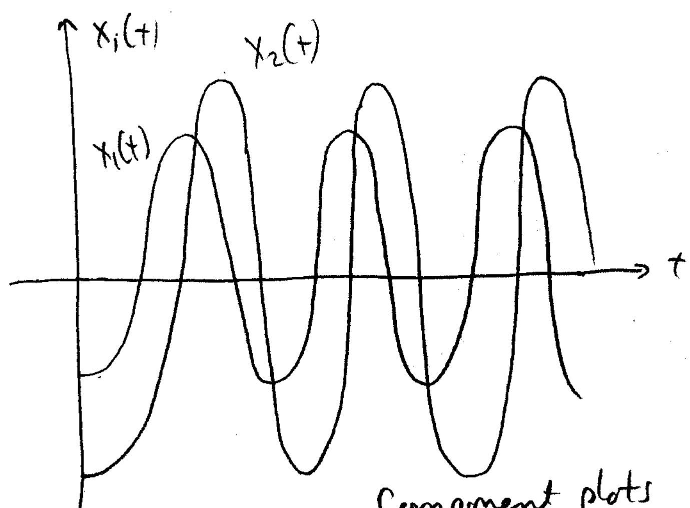
$$\vec{x}(0) = C_1\begin{pmatrix} 5 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \Rightarrow C_1 = -\frac{1}{5}, \quad C_2 = \frac{4}{15}$$

$$\Rightarrow \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -\cos\frac{3}{2}t + \frac{4}{3}\sin\frac{3}{2}t \\ -2\cos\frac{3}{2}t - \frac{2}{3}\sin\frac{3}{2}t \end{pmatrix}$$

This is an ellipse, through $(-1,-2)$ (Use Wolfram Alpha)



Phase portrait



Component plots

Once we have one ellipse drawn (through $(-1,-2)$), the rest are concentric.

Summary of method (for complex eigenvalues)

Suppose we wish to solve $\vec{x}' = A\vec{x}$, $\lambda_{1,2} = a \pm bi$.

We have 2 solutions: $\vec{x}_1(t) = e^{(a+bi)t}\vec{v}_1$ and $\vec{x}_2(t) = e^{(a-bi)t}\vec{v}_2$.

Take one of these (say $\vec{x}_1(t)$), and write it as $\vec{x}_1(t) = \vec{u}(t) + i\,\vec{w}(t)$

(use Euler's formula)

The general solution is $\vec{x}(t) = C_1\vec{u}(t) + C_2\vec{w}(t)$.

The phase plot will be spiraling ellipses, that are

- inward if $a<0$
- outward if $a>0$
- stable if $a=0$

Example 3a: (Repeated eigenvalues, 2 eigenvectors)

Consider the system $\vec{x}' = A\vec{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\vec{x}$.

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1.$$

$(A+I)\vec{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so every vector is an eigenvector.

Pick 2: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We have 2 solutions: $\vec{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{x}_2(t) = e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so the general solution is

$$\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) = C_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

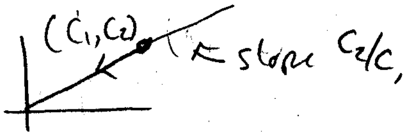
Note: We could have solved this "easy system" the "old way."

$$\begin{cases} x_1' = -x_1 \\ x_2' = -x_2 \end{cases} \Rightarrow \begin{aligned} x_1(t) &= C_1 e^{-t} \\ x_2(t) &= C_2 e^{-t} \end{aligned}$$

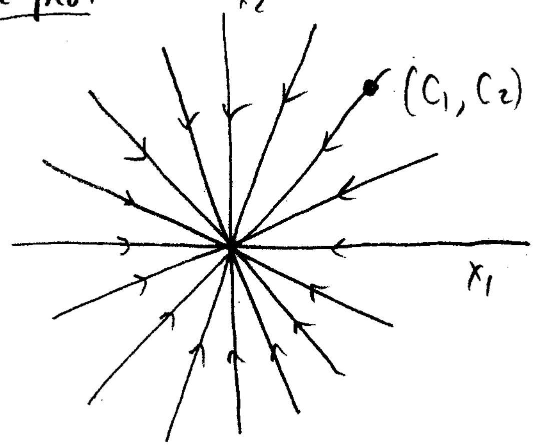
lets plot this: Suppose that $\vec{x}(0)=\begin{pmatrix}c_1\\c_2\end{pmatrix}$. Note that $\dfrac{x_2(t)}{x_1(t)}=\dfrac{c_2e^{-t}}{c_1e^{-t}}=\dfrac{c_2}{c_1}$,

i.e, this solution curve lies on the line with slope $\dfrac{c_2}{c_1}$

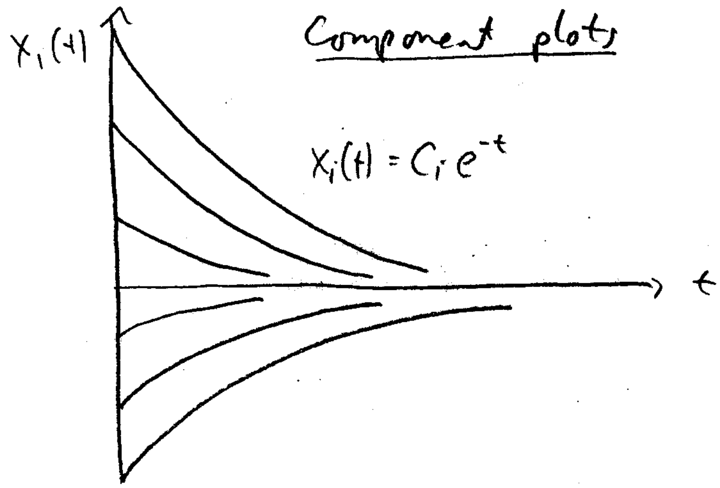
Moreover, $\lim\limits_{t\to\infty}\begin{pmatrix}c_1e^{-t}\\c_2e^{-t}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$



Phase plot



Component plots



$\vec{0}$ is a proper node, or star point.

Example 36: (Repeated eigenvalues, 1 eigenvector)

Consider the system $\vec{x}'=A\vec{x}=\begin{pmatrix}-1&-1\\1&-3\end{pmatrix}\vec{x}$.

$\det(A-\lambda I)=\begin{vmatrix}-1-\lambda&-1\\1&-3-\lambda\end{vmatrix}=(1+\lambda)(3+\lambda)+1=\lambda^2+4\lambda+4=0 \Rightarrow \lambda_1=\lambda_2=-2.$

$(A+2I)\vec{v}=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow x_1-x_2=0 \Rightarrow x_1=x_2$

Thus, $\vec{v}_1=\begin{pmatrix}1\\1\end{pmatrix}$ is the only eigenvector (up to scalar multiple).

We have one solution, $\vec{x}_1(t)=e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}$.

To find a 2nd solution, assume $\boxed{\vec{x}_2(t)=te^{-2t}\vec{v}+e^{-2t}\vec{w}}$,

and solve for $\vec{v}$ & $\vec{w}$.

LHS of $\vec{x}'=A\vec{x}$: $\quad \vec{x}_2{}'=-2te^{-2t}\vec{v}+e^{-2t}(\vec{v}-2\vec{w})$ (product rule)

RHS: $\quad A(te^{-2t}\vec{v}+e^{-2t}\vec{w})$

$$\Rightarrow \; -2te^{-2t}\vec{v} + e^{-2t}(\vec{v} - 2\vec{w}) = Ate^{-2t}\vec{v} + Ae^{-2t}\vec{w}$$

Equate coefficients (vectors) of $\underline{te^{-2t}}$ and $\underline{e^{-2t}}$

$$te^{-2t}: \quad -2\vec{v} = A\vec{v} \quad \Rightarrow (A+2I)\vec{v} = \vec{0} \qquad (1)$$

$$e^{-2t}: \quad \vec{v} - 2\vec{w} = A\vec{w} \quad \Rightarrow (A+2I)\vec{w} = \vec{v} \qquad (2)$$

<u>Note</u>: The solution to (1) is the eigenvector $\vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ of A.

Plug this back into (2):

$$(A+2I)\vec{w} = \begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \begin{cases} w_1 - w_2 = 1 \\ w_1 - w_2 = 1\end{cases}$$

$$\Rightarrow \boxed{w_1 - w_2 = 1} \Rightarrow \vec{w} = \begin{pmatrix}1+c\\c\end{pmatrix}.$$

Since <u>any</u> c works, let's pick one, say $c=0$. $\Rightarrow \vec{w} = \begin{pmatrix}1\\0\end{pmatrix}$

Plug this back into $\vec{x}_2(t) = te^{-2t}\vec{v} + e^{-2t}\vec{w}$:

$$\Rightarrow \vec{x}_2(t) = te^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + e^{-2t}\begin{pmatrix}1\\0\end{pmatrix}$$

Thus, the general solution is $\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t)$

$$= \boxed{C_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-2t}\left[t\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\right]}$$

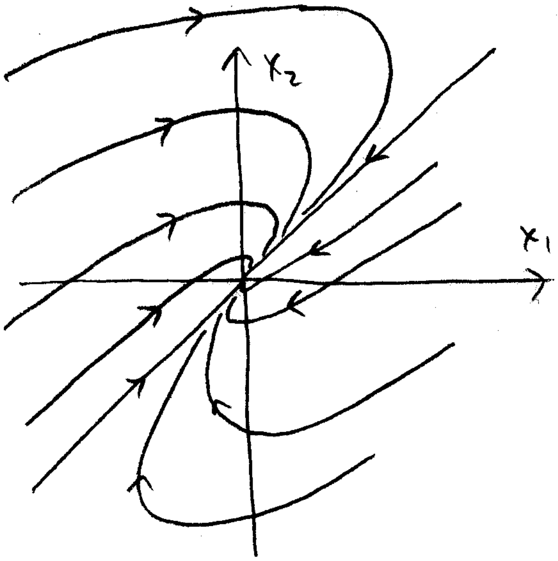
<u>Analyze long-term behavior:</u>

$\lim_{t\to\infty} \vec{x}(t) = \begin{pmatrix}0\\0\end{pmatrix}$. Moreover, $\vec{x}(t) = \overbrace{C_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}}^{\to 0 \text{ faster}} + \overbrace{C_2 e^{-2t}\begin{pmatrix}1\\0\end{pmatrix}}^{\to 0 \text{ faster}} + \overbrace{C_2 t e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}}^{\to 0 \text{ slower}}$

Thus, the $te^{-2t}\begin{pmatrix}1\\1\end{pmatrix}$ term "dominates" the limit as $t \to +\infty$

Similarly, the $te^{-2t}\begin{pmatrix}1\\1\end{pmatrix}$ term dominates the limit as $t \to -\infty$, so forward <u>and</u> backwards in time, solutions become more parallel to $\vec{v} = \begin{pmatrix}1\\1\end{pmatrix}$.

Phase portrait

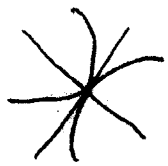


* As before, we get a solution lying on the "eigenvector line" $\vec{v}_1 = \binom{x_1}{x_2}$, (because it has slope x_2/x_1).

* The point $\binom{0}{0}$ is called an improper, or degenerate node.

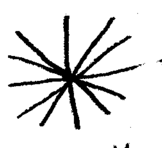
Summary of phase portraits / stability of $\vec{x}' = A\vec{x}$, $\det A \neq 0$.

real, distinct $\begin{cases} \lambda_1 > \lambda_2 > 0 \\ \lambda_1 < \lambda_2 < 0 \\ \lambda_1 < 0 < \lambda_2 \end{cases}$



$\lambda_1 < 0 < \lambda_2 \longrightarrow$ "saddle" $\longrightarrow$

repeated $\left\{ \lambda_1 = \lambda_2 \right.$



"star"

or

"improper node"

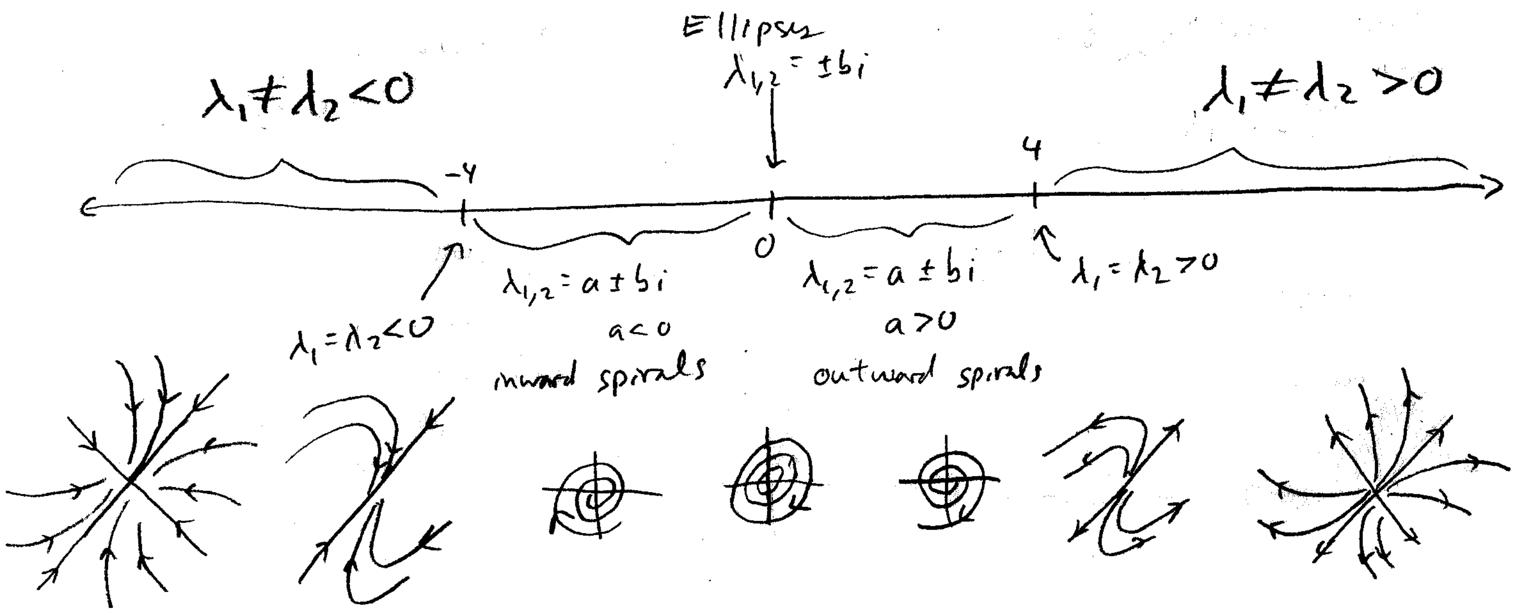
complex $\begin{cases} \lambda_{1,2} = a \pm bi & \text{Spirals/ellipses.} \\ a > 0 & \text{outward} \\ a < 0 & \text{inward} \\ a = 0 & \text{concentric} \end{cases}$

Let's see how the phase portrait changes as λ changes.

<u>Example</u>: Consider $\vec{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \vec{x}$, where α is a parameter.

Check: $|A - \lambda I| = \lambda^2 - \alpha\lambda + 4 = 0 \Rightarrow \boxed{\lambda_{1,2} = \dfrac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}}$

* Let's vary α between $\pm\infty$. Note: $\alpha = \pm 4$ yields a double root.



Remark: For no value of α is the phase portrait a saddle!

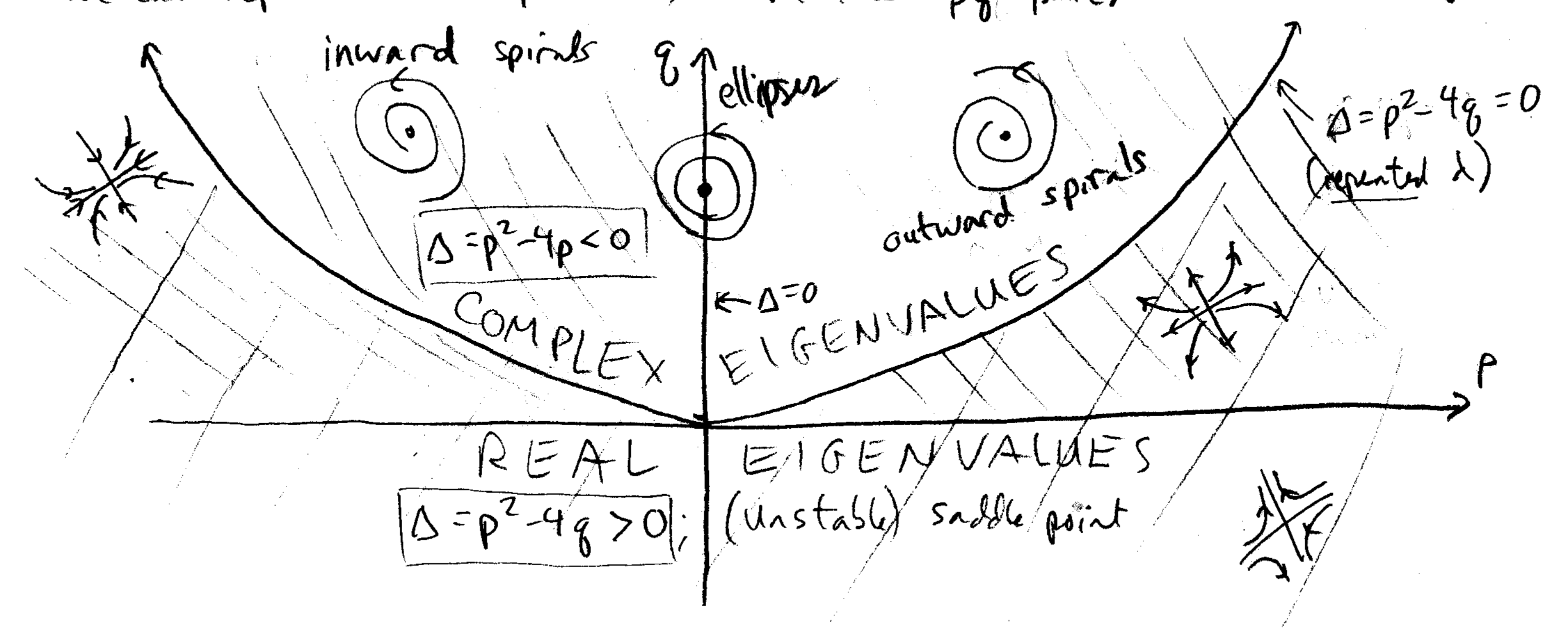
Actually, this line is a cross-section of a larger picture, capturing all possible eigenvalues & phase portraits.

Suppose that the characteristic equation of A is

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - p\lambda + q = 0$$

Then the eigen values of A are $\lambda = \dfrac{p \pm \sqrt{p^2-4q}}{2a}$.

We can represent all possible λ's on the pq-plane; Let $\Delta = p^2 - 4q$



A word on higher-order systems of ODEs

Basically, the same ideas carry over, but the math is more complicated for $n \times n$ matrices, $n > 2$.

SIR model: (A popular 3×3 system of ODEs).

This models an epidemic disease in a population (e.g., flu).

Let $S(t)$ = # susceptible people at time t

$I(t)$ = # infected people at time t.

$R(t)$ = # recovered (immune) people at time t.

Initially, there are N susceptible (uninfected) people.

People transition: Susceptible $\longrightarrow$ Infected $\longrightarrow$ Recovered

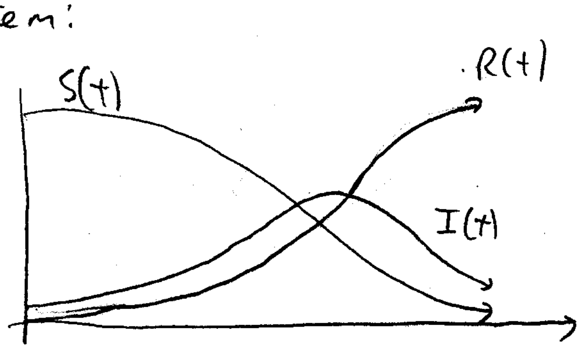
$$\frac{dS}{dt} = -aSI \quad \text{"proportional to both } S(t) \text{ \& } I(t)\text{"}$$

$$\frac{dI}{dt} = aSI - bI \quad \text{"(rate people get sick) - (rate people get healthy)"}$$

$$\frac{dR}{dt} = bI \quad \text{"rate people get healthy"}$$

We get a nonlinear, autonomous system:

$$\begin{cases} S' = -aSI & S(0) = N \\ I' = aSI - bI & I(0) = 1 \\ R' = bI & R(0) = 0 \end{cases}$$



Component plots.