MthSc 208: Differential Equations (Fall 2010)
In-class Worksheet 20: The Wave Equation

NAME:

We will solve for the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the wave equation:

$$u_{tt} = c^2 u_{xx} \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = x(\pi - x), \qquad u_t(x,0) = 1\,.$$

(a) Carefully descsribe (and sketch) a physical situation that this models.

(b) Assume that $u(x,t) = f(x)g(t)$. Compute u_t, u_{tt}, u_x, u_{xx}, and find boundary conditions for $f(x)$.

(c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ, and write down two ODEs: one for $f(x)$ and one for $g(t)$.

(d) Solve the ODE for $f(x)$ (including the boundary conditions), and determine λ. You may assume that $\lambda = -\omega^2 < 0$.

(e) Now that you know what λ is, solve the ODE for $g(t)$.

(f) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(x,t) = f_n(x)g_n(t)$.

(g) Find the particular solution to the initial value problem by using the initial conditions. The following information is useful:

The Fourier sine series of $x(\pi - x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{4}{\pi n^3}(1-(-1)^n)\sin nx$.

(h) What is the long-term behavior of the system? Give a mathematical, and physical, justification.