MthSc 412, Fall 2010

HW #1. Due Tuesday, August 24th.

- Read VGT, Chapters 1 & 2.
- VGT Exercises 1.2, 1.8–1.13.

HW #2. Due Tuesday, August 31st.

- Read VGT, Chapter 3.
- VGT Exercises 1.14, 2.2, 2.6, 2.9, 2.11, 2.12, 2.17, 2.19.
- Draw a Cayley diagram for the "square puzzle group" using 2 generators that are both reflections. Compare this to the Cayley graph using a rotation and a reflection as the generating set (which we did in class; see Exercise 2.8).
- Do Exercise 2.18. Compare your Cayley diagram for this problem to the one in Exercise 2.5, which we did in class. What can you say about these two groups?

HW #3. Due Tuesday, September 6th.

- Read VGT, Chapter 4.
- VGT Exercises 3.8, 3.11, 3.13-16.

HW #4. Due Friday, September 10th.

- Read AATA, Chapter 3.
- VGT Exercises 4.1, 4.2, 4.15, 4.19-24.

HW #5a. Due Wednesday, September 15th.

- Read VGT, Chapter 5.
- VGT Exercises 5.1, 5.2(d,e,f), 5.3 (give a *brief* justification for each true/false), 5.10(a,b,c,d), 5.11-14.

HW #5b. Due Tuesday, September 21th.

- Read AATA, Chapters 5 and 6.
- VGT Exercises 5.15(e,f), 5.20, 5.30, 5.34, 5.35, 5.37, 5.41(b), 5.42.

HW #6a. Due Tuesday, September 27th

- Read VGT, Chapter 6. Read AATA, Chapter 6.
- VGT Exercises 6.5, 6.7, 6.9, 6.12, 6.17, 6.28.
- Prove that every subgroup of a cyclic group is cyclic. (Do not assume that G is finite).

HW #6b. Due Tuesday, October 5th.

- VGT Exercises 6.11, 6.18, 6.20, 6.29.
- VGT Exercise 6.22. Additionally, construct the subgroup lattice, or Hasse diagram, for the following groups: V_4, C_4, C_{24}, D_4, and label each edge with the corresponding index.
- Prove the following (do not refer to Cayley diagrams):
  (a) If $\mathcal{H}$ is a collection of subgroups of G, then the intersection $\bigcap_{H\in\mathcal{H}} H$ is a subgroup of G.
  (b) If $S \subset G$, then $\langle S\rangle$ is the intersection of all subgroups containing S. [*Hint*: One way to prove that $A = B$ is to show that $A \subset B$ and $B \subset A$.]
- (a) Prove that if $x \in H$, then $xH = H$. What is the interpretation of this statement in terms of the Cayley diagram?
  (b) Prove, that if $b \in aH$, then $aH = bH$. (Use the definition of a coset: $aH = \{ah \colon h \in H\}$.)

HW #7a. Due Tuesday, October 12th.

- Read VGT, Chapter 7.1–3.
- VGT Exercises (Products): 7.7, 7.8, 7.12, 7.13
- VGT Exercises (Quotients): 7.17, 7.18(c,d,e,f,g,h), 7.24.
- Recall that G/H is the set of (left) cosets of H in G. We defined a binary operation on G/H of left cosets by $aH \cdot bH = abH$. In this exercise, you will see further motivation for

this definition. Given $a, b \in G$, define the sets

$$aHbH = \{ah_1bh_2 \colon h_1, h_2 \in H\} \qquad \text{and} \qquad abH = \{abh \colon h \in H\}.$$

Prove that if $H \lhd G$, then $aHbH = abH$ (show that an arbitrary element of abH is in $aHbH$, and vice-versa). Comment on how this relates to quotient groups.

- Prove or disprove the following statements (without refering to Cayley diagrams).
  (a) Every subgroup of an abelian group is normal.
  (b) Every quotient of an abelian group is abelian.
  (c) If $K \lhd H \lhd G$, then $K \lhd G$.

HW #7b. Due Tuesday, October 19th.

- Read VGT, Chapter 7.3–4. Read AATA, Chapters 9.2, 10.1.
- VGT Exercises (Normalizers): 7.25(c,d), 7.26(c,d), 7.27.
- VGT Exercises (Conjugacy): 7.29, 7.32, 7.33.
- Recall that the *center* of a group G is the set $Z(G) = \{z \in G \colon gz = zg \ \forall g \in G\}$.
  (a) Prove that if $G/Z(G)$ is cyclic, then G is abelian.
  (b) What group must $G/Z(G)$ be, in this case?
- Write the following permutations in cycle notation, as a product of disjoint cycles.
  (a) $(1\ 2)\,(1\ 2\ 3) \in S_3$
  (b) $(1\ 2\ 3)\,(1\ 2) \in S_3$
  (c) $(1\ 3\ 2)\,(1\ 2)\,(1\ 4\ 2) \in S_4$
  (d) $(3\ 4)\,(1\ 2\ 3\ 4\ 5\ 6)\,(3\ 4) \in S_6$

HW #8a. Due Friday, October 22nd.

- Read VGT, Chapter 8. Read AATA, Chapters 9, 10, 12.1
- VGT Exercises (Embeddings & quotient maps): 8.2–5. 8.8, 8.10(c,d,e,f), 8.12, 8.15–17, 8.36. (For 8.2 give a *brief* justification for each true/false.)
  (a) Prove that if $H < G$, then $H \cong gHg^{-1}$, for any $g \in G$. (Recall that we showed in class that gHg^{-1} is always a subgroup of G.)
  (b) Use Part (a) to show that in any group, $|xy| = |yx|$.

HW #8b. Due Wednesday, November 4th.

- Read AATA, Chapters 9, 10, 12.1
- VGT Exercises (FHT): 8.13, 8.14, 8.40(book has a typo, see online errata).
- VGT Exercises (Modular arithmetic): 8.20, 8.22, 8.23.
- VGT Exercises (Finite abelian groups): 8.50.
- VGT Exercises (Misc.): 8.39(a), 8.41–43.
- For each order, list all abelian groups of that order (up to isomorphism), as a product of cyclic groups of prime-power order.
  (a) $32 = 2^5$
  (b) $60 = 2^2 \cdot 3 \cdot 5$
  (c) $108 = 2^2 \cdot 3^3$

  An alternative way to write a finite abelian group is

  $$A \cong C_{k_1} \times C_{k_2} \times \cdots \times C_{k_\ell},$$

  where $k_1 \mid k_2 \mid \cdots k_{\ell-1} \mid k_\ell$ (but the k_i's need not be prime powers). For each order in the previous question, list all abelian groups of that order in this manner.
- Prove that if A and B are normal subgroups of G, and $AB = G$, then

  $$G/(A \cap B) \cong (G/A) \times (G/B).$$

  [*Hint*: Construct a homomorphism $\phi \colon AB \to (G/A) \times (G/B)$ that has kernel $A \cap B$, then apply the FHT.]
- Let A and B be normal subgroups of G.
  (a) Prove that AB is a subgroup of G.
  (b) Prove that $B \lhd AB$ and $A \cap B \lhd A$.

(c) Prove that $A/(A \cap B) \cong AB/B$. [*Hint:* Construct a homomorphism $\phi\colon A \to AB/B$ that has kernel $A \cap B$, then apply the FHT.]
(d) Draw a diagram, or lattice, of G and its subgroups AB, A, B, and $A \cap B$. Interpret the result in Part (c) in terms of this diagram.

HW #9a. Due Friday, November 12th.

- Read VGT, Chapters 9.1, 9.2, 9.3.
- VGT Exercises 9.3, 9.4, 9.7, 9.9, 9.10, 9.12.
- Repeat the exercise from the class lecture notes for several other groups: Let S be the set of length-4 binary necklaces. Draw an action diagram for each of the following group actions:
  (a) $\phi\colon V_4 \to \mathrm{Perm}(S)$, where $\phi(h)$ reflects each necklace about a vertical axis, and $\phi(v)$ reflects each necklace about a horizontal axis;
  (b) $\phi\colon C_4 \to \mathrm{Perm}(S)$, where $\phi(1)$ rotates each necklace 90° clockwise;
  (c) $\phi\colon D_4 \to \mathrm{Perm}(S)$, where $\phi(r)$ rotates each necklace 90° clockwise, and $\phi(f)$ reflects each necklace about a vertical axis.

  Additionally, pick an element in each orbit and find its stabilizer.
- Let G be a group of order 15, which acts on a set S with 7 elements. Show that the group action has a fixed point.
- Let G act on itself (i.e., $S = G$) by conjugation.
  (a) Show that the set of fixed points of this action is $Z(G)$, the center of G.
  (b) Prove that if G is a p-group (i.e., $|G| = p^n$ for some prime p), then $Z(G)$ is nontrivial.
  (c) Use Part (b) to completely classify all simple p-groups.

HW #9b. Due Friday, November 19th.

- Read VGT, Chapters 9.4, 9.5. Read AATA, Chapters 13, 14.
- VGT Exercises 9.17, 9.21, 9.22, 9.23
- Prove that a Sylow p-subgroup of G is normal if and only if it is the unique Sylow p-subgroup of G.
- Recall that a group G is called *simple* if its only normal subgroups are G and $\{e\}$.
  - Show that there is no simple group of order pq, where $p < q$ and are both prime.
  - Show that there is no simple group of order $108 = 2^2 \cdot 3^3$.
- Suppose that $H \lneq G$, and let $S = G/H$. Let G act on S, where $\phi(g)\colon xH \mapsto gxH$.
  (a) Show that if $|G|$ does not divide $[G : H]!$, then G cannot be simple.
  (b) Use (a), together with the Sylow theorems, to show that any group of order 36 cannot be simple.

HW #10a. Due Monday, November 29th.

- Read VGT, Chapters 10.1, 10.2, 10.3, 10.4, 10.5. Read AATA, Chapters 20.1, 20.2, 22.1.
- VGT Exercises 10.8, 10.14, 10.15, 10.18, 10.19, 10.22, 10.29(book has a typo, see online errata), 10.30.
- Without using a calculator or computer, determine if the following polynomials are irreducible. If not, factor them into irreducible factors.
  (a) $x^4 - 3x^3 + 12x^2 + 51$
  (b) $2x^3 - 12x^2 + 20x + 4$
  (d) $60x^2 + 50x - 10$
  (c) $x^4 + 7x^2 + 10$

HW #10b. Due Monday, December 6th.

- Read VGT, Chapters 10.6, 10.7. Read AATA, Chapters 22.2, 22.3.
- VGT Exercises 10.3, 10.13, 10.16, 10.20, 10.26.
- Let $f(x) = x^4 - 7$.
  (a) Determine the splitting field F of f. Give an explicit basis. What does this tell you about the order of the Galois group G of f?
  (b) Compute the Galois group of f. Write down the two automorphisms that generate it (call them r and f).

(c) Draw the subgroup lattice of G. Label the edges by index, and circle the subgroups that are normal in G.
(d) Draw the subfield lattice of F. Label the edges by degree, and circle the subfields that are normal extensions of $\mathbb{Q}$.
(e) For each intermediate subfield K, write down the largest subgroup of G that fixes K.
(f) For each subgroup $H < G$, write down the largest intermediate subfield fixed by H.

- The roots of the polynomial $f(x) = x^n - 1$ are called the n^{th} *roots of unity*.
  (a) For $n = 3, \ldots, 8$, sketch the roots of $x^n - 1$ in the complex plane.
  (b) For each $n = 3, \ldots, 8$, write $x^n - 1$ as a product of irreducible factors. [*Hint*: Try googling *cyclotomic polynomial*.]
  (c) The Galois group of $x^n - 1$ is the group U_n, or $(\mathbb{Z}/n\mathbb{Z})^\times$ (see Exercise 8.41). Justify this by explicitly describing the automorphisms of the splitting field that generate the Galois group.