MthSc 412: Abstract Algebra

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Introduction

1. Welcome to MthSc 412, Fall 2010!
2. Introductions
3. Discussion of the class, and the syllabus
4. Expectations and general game plan
5. My web page

   http://www.math.clemson.edu/~macaule/

6. Web page for textbooks (including errata)

   http://web.bentley.edu/empl/c/ncarter/vgt/
   http://abstract.ups.edu/

7. Group Explorer

   http://groupexplorer.sourceforge.net/

8. OK, let's get started!

Chapter 1: What is a group?

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Our introduction to group theory will begin by discussing the famous Rubik's Cube.

It was invented in 1974 by Ernő Rubik of Budapest, Hungary

Ernő Rubik is a Hungarian inventor, sculptor and professor of architecture.

According to his Wikipedia entry:

> *He is known to be a very introverted and hardly accessible person, almost impossible to contact or get for autographs.*

Not impossible . . . almost impossible.

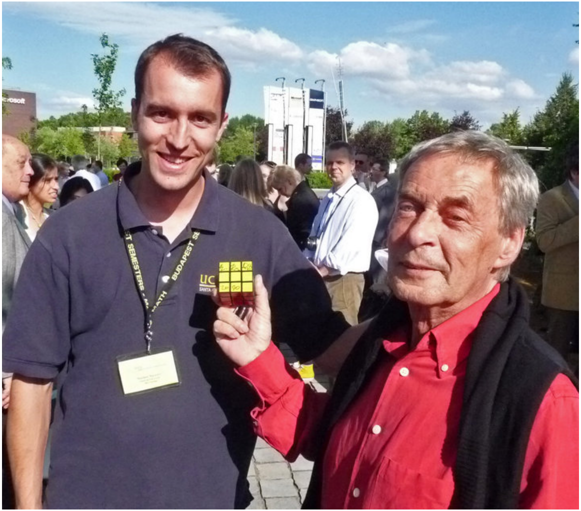
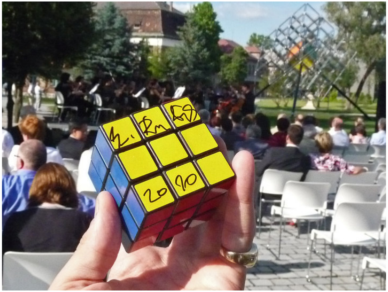
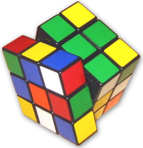


Figure: June 2010, in Budapest, Hungary

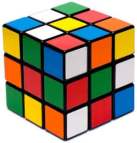
- The cube comes out of the box in the solved position:



- But then we can scramble it up by consecutively rotating one of its 6 faces:



- The result might look something like this:



- The goal is to return the cube to its original solved position, again by consecutively rotating one of the 6 faces.

Since Rubik's Cube does not seem to require any skill with numbers to solve it, you may be inclined to think that this puzzle is not mathematical.

Group theory is not primarily about numbers, but rather about patterns and symmetry; something the Rubik's Cube possesses in abundance.

Let's explore the Rubik's Cube in more detail. In particular, let's identify some key features that will be recurring themes in our study of patterns and symmetry.

First, some questions to ponder:

- How did we scramble up the cube in the first place? How do we go about unscrambling the cube?
- In particular, what actions, or moves, do we *need* in order to scramble and unscramble the cube? (There are many correct answers.)
- How is Rubik's Cube different from checkers?
- How is Rubik's Cube different from poker?

Let's make 4 key observations:

Observation 1.1

There is a predefined list of moves that never changes.

Observation 1.2

Every move is reversible.

Observation 1.3

Every move is deterministic.

Observation 1.4

Moves can be combined in any sequence.

We could add more to our list, but as we shall see, these 4 observations are sufficient to describe the aspects of the mathematical objects that we wish to study.

Group theory studies the mathematical consequences of these 4 observations, which in turn will help us answer interesting questions about symmetrical objects.

Group theory arises everywhere! In puzzles, visual arts, music, nature, the physical and life sciences, computer science, cryptography, and of course, all throughout mathematics.

Group theory is arguably one of the most beautiful subjects in all of mathematics!

Rules of a group

Instead of considering our 4 observations as descriptions of Rubik's Cube, let's rephrase them as rules (axioms) that will define the boundaries of our objects of study.

Advantages of our endeavor:

1. We make it clear what it is we want to explore.
2. It helps us speak the same language, so that we know we are discussing the same ideas and common themes, though they may appear in vastly different settings.
3. The rules provide the groundwork for making logical deductions, so that we can discover new facts (many of which are surprising!).

Rules of a group

Our rules:

Rule 1.5

There is a predefined list of actions that never changes.

Rule 1.6

Every action is reversible.

Rule 1.7

Every action is deterministic.

Rule 1.8

Any sequence of consecutive actions is also an action.

Rules of a group

What changes were made in the rephrasing?

Comments

- We swapped the word *move* for *action*.
- The (usually short) list of actions required by Rule 1.5 is our set of building blocks; called the generators.
- Rule 1.8 tells us that any sequence of the generators is also an action.

Finally, here is our unofficial definition of a group. (We'll make things a bit more rigorous later.)

Definition 1.9

A group is a system or collection of actions satisfying Rules 1.5–1.8.

Observations about the "Rubik's Cube group"

Frequently, two sequences of moves will be "indistinguishable." We will say that two such moves are *the same*. For example, rotating a face (by 90°) once has the same effect as rotating it five times.

Fact: There are 43,252,003,274,489,856,000 distinct configurations of the Rubik's cube.

While there are infinitely many possible sequences of moves, starting from the solved position, there are 43,252,003,274,489,856,000 "truly distinct" moves.

All 4.3×10^{19} moves are generated by just 6 moves: a 90° twist of one of the 6 faces.

Let's call these generators a, b, c, d, e, and f. Every word over the alphabet $\{a, b, c, d, e, f\}$ describes a unique configuration of the cube (starting from the solved position).

Group Exercises

Let's explore a few more examples.

1. Discuss Exercise 1.1 (see Bob = Back of book) as a large group.
2. In groups of 2–3, complete the following exercises (not collected):
   - Exercise 1.3 (see Bob)
   - Exercise 1.4
3. I'd like two groups to volunteer to discuss their answers to the two previous exercises.
4. Now, mix the groups up, so that no group stays the same. In your new groups, complete Exercise 1.8. I want each group to write up a complete solution.