We will solve for the function $u(x,y,t)$ defined for $0 \leq x, y \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the heat equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
\left. u \right|_{t=0} &= 2 \sin x \sin 2y + 3 \sin 4x \sin 5y.
\end{align*}
\]

(a) Carefully describe (and sketch) a physical situation that this models.

(b) Assume that $u(x,y,t) = f(x,y)g(t)$. Compute $u_{xx}$, $u_{yy}$, and $u_t$, find boundary conditions for $f(x,y)$. 

\[
\begin{align*}
\left. u \right|_{t=0} &= 2 \sin x \sin 2y + 3 \sin 4x \sin 5y.
\end{align*}
\]
(c) Plug \( u = fg \) back into the PDE and separate variables by dividing both sides of the equation by \( c^2fg \). Set this equal to a constant \( \lambda \), and write down two equations: an ODE for \( g(t) \), and a PDE \( f(x, y) \) (called the Helmholtz equation), with four boundary conditions.

(d) Solve the ODE for \( g(t) \).

(e) To solve the PDE for \( f \), assume that \( f(x, y) = X(x)Y(y) \). Plug this back in and separate variables. [For consistency, put the \( X''/X \) term on one side of the equation, and set equal to a constant \( \mu \).]
(f) Write down two ODEs – one for $X(x)$ and one for $Y(y)$, and include boundary conditions for both.  

*Hint:* It is easier notationally if you introduce a new constant, $\nu := \lambda - \mu$.

(g) Solve the ODEs for $X(x)$ and $Y(y)$, and determine $\mu$ and $\nu$ (and hence $\lambda$). You should get a $\lambda$ for each choice of positive integers $n, m \in \mathbb{N}$, call it $\lambda_{nm}$.

(h) For each $n, m \in \mathbb{N}$, we have a solution $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$. Write down this solution.
(i) Find the general solution of the PDE. It will be a doubly infinite sum (superposition) of solutions:
\[ \sum_{n,m \in \mathbb{N}} u_{nm}(x, y, t). \]

(g) Find the particular solution to the initial value problem by using the initial condition.

(h) What is the long-term behavior of the system? Give a mathematical, and physical, justification.