1. For each vector \( \mathbf{v} \), compute its norm, \( ||\mathbf{v}|| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \), and then normalize it, by computing \( \mathbf{v}/||\mathbf{v}|| \).

\[
\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.
\]

2. For a unit vector \( \mathbf{n} \), the projection of \( \mathbf{v} \) onto \( \mathbf{n} \) is the quantity \( \mathbf{v} \cdot \mathbf{n} \). This measures the magnitude of \( \mathbf{v} \) in the \( \mathbf{n} \)-direction. Consider the following four unit vectors:

\[
\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{n}_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \quad \mathbf{n}_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}.
\]

(a) Draw the vectors \( \{\mathbf{e}_1, \mathbf{e}_2\} \) in \( \mathbb{R}^2 \), and sketch the square “grid” that they determine. Do the same thing for \( \{\mathbf{n}_1, \mathbf{n}_2\} \) but on a new set of axes.

(b) Write the vector \( \mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) as a linear combination of \( \{\mathbf{e}_1, \mathbf{e}_2\} \). That is, write \( \mathbf{w} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 \) and determine \( a_1, a_2 \in \mathbb{R} \). Then, write \( \mathbf{w} \) as a linear combination of \( \{\mathbf{n}_1, \mathbf{n}_2\} \).

(c) Sketch \( \mathbf{w} \) on both sets of axes, and show how these sketches match your answers to Part (b).

(d) The \( 2 \times 2 \) matrix \( \mathbf{A} = [\mathbf{n}_1 \ \mathbf{n}_2] \) can be thought of as a linear map, \( \mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2 \). Describe this linear map (geometrically) in a sentence. [Hint: You can think of the “input” as one of your grids, and the “output” as the other grid.]

3. A set \( \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \) of vectors is orthogonal (or perpendicular) if \( \mathbf{v}_i \cdot \mathbf{v}_j = 0 \) for all \( i \neq j \). The set is furthermore orthonormal if each \( \mathbf{v}_i \) is a unit vector. That is, if

\[
\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}
\]

(a) Show that the set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \), where

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}
\]

is an orthogonal set, but not orthonormal.

(b) Normalize \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) to get an orthonormal basis of \( \mathbb{R}^3 \), \( \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} \).

(c) Express the vector \( \mathbf{w} = (1, 2, 3) \) in terms of \( \mathbf{n}_1, \mathbf{n}_2, \) and \( \mathbf{n}_3 \). That is, find \( a_1, a_2, \) and \( a_3 \) such that

\[
\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a_1 \mathbf{n}_1 + a_2 \mathbf{n}_2 + a_3 \mathbf{n}_3.
\]
(d) Express the vector \( w \) as a linear combination of \( v_1, v_2, \) and \( v_3 \). That is, write 
\[
    w = b_1 v_1 + b_2 v_2 + b_3 v_3
\]
and find a formula for each \( b_i \). [Hint: It should be in entirely in terms of dot products of \( v_i \) and \( w \). Start by substituting \( v_i/||v_i|| \) in for \( n_i \) in your answer to Part (c).]

4. Consider the matrix 
\[
A = \begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}.
\]

(a) Find a non-zero linear combination \( x_1 u + x_2 v + x_3 w \) of the column vectors of \( A \) that gives the zero vector.

(b) Describe the set of all solutions to \( x_1 u + x_2 v + x_3 w = 0 \). [Hint: It is a line. Which line is it?]

(c) Describe the set of all linear combinations \( x_1 u + x_2 v + x_3 w \). We say that this is the subspace of \( \mathbb{R}^3 \) that is spanned by the set \( \{u, v, w\} \).

5. Given an \( n \times m \) matrix \( A = [a_{ij}] \), the transpose of \( A \) is an \( m \times n \) matrix defined as \( A^T = [a_{ji}] \). For each of the following three matrices \( M \), compute its transpose \( M^T \), as well as the products \( M^T M \) and \( MM^T \).

\[
A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

6. Consider the following vectors \( u, v, w \), and the matrix \( A = [u \ v \ w] \):

\[
    u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad A = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}.
\]

(a) Write out \( A^T A \) in terms of the dot products of \( u, v \) and \( w \).

(b) Write out \( A^T A \) and in terms of the vectors \( u, v \) and \( w \) and their transposes, but not their individual entries. [Recall that \( u \cdot v = u^T v \) .]

(c) Give a complete characterization of which matrices \( A \) have the property that \( A^T A = I \), where \( I \) is the identity matrix. Give a geometric description of your answer in terms of the column vectors.