

Read: Strang, Section 1.3, 2.1, 2.2, 2.3.

1. In this problem, we will solve a system $\mathbf{Ax} = \mathbf{b}$ using the method of *elimination*, also called *row-reduction*. The system is shown in matrix form on the left, and the corresponding *augmented matrix* is shown at right, which is a useful notation.

$$\begin{bmatrix} 1 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad [\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{cc|c} 1 & -2 & 3 \\ -4 & 2 & 0 \end{array} \right].$$

Carry out the following steps for this system.

- Perform a single elimination step to convert the equation $\mathbf{Ax} = \mathbf{b}$ into an equation $\mathbf{Ux} = \mathbf{c}$, where \mathbf{U} is an *upper triangular matrix*. What is the corresponding elementary matrix, \mathbf{E}_{21} ? Notice that $\mathbf{U} = \mathbf{E}_{21}\mathbf{A}$.
 - Now, perform a single elimination step to remove the entry of \mathbf{U} above the main diagonal. You will end up with a equation of the form $\mathbf{Dx} = \mathbf{d}$, where \mathbf{D} is a *diagonal matrix*.
 - What is \mathbf{D}^{-1} , the *inverse matrix* of \mathbf{D} ? (You should be able to do this by inspection.) Multiply both sides of the equation $\mathbf{Dx} = \mathbf{d}$ by \mathbf{D}^{-1} to solve for \mathbf{x} .
 - The solution you obtained in Part (c) is actually the vector $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Express \mathbf{A}^{-1} as a product of matrices that you found above (\mathbf{E}_{12} , \mathbf{E}_{21} , and \mathbf{D}^{-1}).
2. Consider the following 3×3 system, $\mathbf{Ax} = \mathbf{b}$. The augmented matrix $[\mathbf{A} \mid \mathbf{b}]$ is not shown.

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \\ -2 & -7 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

- Perform a sequence of elimination steps to convert the equation $\mathbf{Ax} = \mathbf{b}$ into an equation $\mathbf{Ux} = \mathbf{c}$, where \mathbf{U} is an *upper triangular matrix*. For each step, write down the corresponding elementary matrix, \mathbf{E}_{ij} .
 - Express \mathbf{U} as a product of elementary matrices and \mathbf{A} . (Note that \mathbf{A} should be the rightmost matrix in the expression.)
 - Continue the elimination process to eliminate the entries of \mathbf{U} above the main diagonal. You will end up with a equation of the form $\mathbf{Dx} = \mathbf{d}$, where \mathbf{D} is a diagonal matrix.
 - Find \mathbf{D}^{-1} and multiply it to both sides of the equation $\mathbf{Dx} = \mathbf{d}$ to solve for \mathbf{x} .
 - Note that the solution you obtained in Part (d) is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Express \mathbf{A}^{-1} as a product of elementary matrices and \mathbf{D}^{-1} .
3. In both of the following equations, the matrix \mathbf{A} is not invertible, so there will either be no solution, or infinitely many solutions to the equation $\mathbf{Ax} = \mathbf{b}$.

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Carry out the following steps for each equation above.

- (a) Graph the two linear equations of the system (this is the “row picture”).
- (b) Use the elimination method until you get an unavoidable zero in a pivot location, in which case you have to stop. At this point you will have an equation $\mathbf{U}\mathbf{x} = \mathbf{c}$, where \mathbf{U} is upper-triangular.
- (c) Write out the two equations that the system $\mathbf{U}\mathbf{x} = \mathbf{c}$ represents. Are there no solutions, or infinitely many solutions? Why?

4. We will solve the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ for each of the three matrices \mathbf{A} shown below.

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 5 & 6 \\ 5 & -4 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 3 & -6 & 3 \end{bmatrix}$$

Recall that if \mathbf{A} is not invertible, then there will be a non-zero solution to the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$. Carry out the following steps for each equation above.

- (a) Use the elimination method to row-reduce \mathbf{A} to an upper-triangular matrix \mathbf{U} , and if possible, to a diagonal matrix \mathbf{D} . (Show your steps, but you do not need to write out each individual elementary matrix like you did in Problems 1 and 2.)
 - (b) Describe the set of all solutions to $\mathbf{U}\mathbf{x} = \mathbf{0}$ (and hence to $\mathbf{A}\mathbf{x} = \mathbf{0}$) as best you can. Give a geometric interpretation (e.g., a point, line, plane, etc.)
5. Find the inverse of the following matrices by row-reducing the corresponding *augmented matrix* until the left half is the identity matrix.

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & -7 & 0 & 1 \end{array} \right]$$

$$[\mathbf{B} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 3 & 1 & 4 & 1 & 0 & 0 \\ 5 & 2 & 6 & 0 & 1 & 0 \\ -7 & -2 & -9 & 0 & 0 & 1 \end{array} \right]$$

When you do this, the augmented matrices that you end up with will be $[\mathbf{I} \mid \mathbf{A}^{-1}]$ and $[\mathbf{I} \mid \mathbf{B}^{-1}]$, respectively. [*Hint*: The matrix \mathbf{B}^{-1} has only integer entries – by scaling appropriately, you can avoid fractions altogether in the elimination process.]